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LIQUID PROPELLANT SLOSHING IN MOBILE TANKS OF ARBITRARY SHAPE

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ABSTRACT

This report considers the irrotational motion of an incompressible, inviscid liquid contained in mobile tanks of arbitrary shape. Hydrodynamic equations are derived for six degrees of freedom. All quantities are written in terms of a coordinate system which moves with the tank. The pressure, forces, moments, and surface wave height are all obtained in terms of nondimensional parameters. For tanks with an axis of symmetry and three degrees of freedom, these equations are matched with corresponding equations of motion of two mechanical systems: spring-mass and pendulum.

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LIST OF THEOREMS

$$\int_V \frac{\partial A_j}{\partial x_i}\, dV = \int_S A_j \nu_i\, dS \tag{I}$$

$$\int_V \frac{\partial u}{\partial x_i} w_i\, dV = \int_S (u w_i) \nu_i\, dS - \int_V u \frac{\partial w_i}{\partial x_i} \tag{II}$$

$$\int_V \left(u \frac{\partial^2 w}{\partial x_i \partial x_i} - w \frac{\partial^2 u}{\partial x_i \partial x_i}\right) dV = \int_S \left(u \frac{\partial w}{\partial x_i} - w \frac{\partial u}{\partial x_i}\right) \nu_i\, dS \tag{III}$$

$$\int_{US} (x_\alpha \nu_\beta - x_\beta \nu_\alpha)\, \psi_i\, dS = \int_{US} \left[\nu_k x_\alpha x_\beta \frac{\partial \psi_i}{\partial x_k} - 2 x_\beta \nu_\alpha \psi_i\right] dS \tag{IV}$$

$$\int_{UV} \epsilon_{ijk} x_j \frac{\partial \psi_m}{\partial x_k}\, dV = \int_{US} \epsilon_{ijk} x_j \nu_k \psi_m\, dS \tag{V}$$

NOMENCLATURE

y_i	Rectangular Cartesian coordinates, fixed in space
$y_i^\star$	Rectangular Cartesian coordinates, translating in space
x_i	Rectangular Cartesian coordinates, translating and rotating in space
a_{ij}	Direction cosines = $\cos(x_i, y_j)$
δ_{ij}	Kronecker delta
ϵ_{ijk}	Alternating tensor
ω_i	Components of angular velocity in the x_i-system
q_i	Components of velocity of a particle in the x_i-system
u_i	Components of translational velocity of the origin of the x_i-system, measured in the x_i-system
ρ	Fluid density
p	Hydrodynamic pressure
F_i	Components of the body force in the x_i-system
a_i	Components of acceleration of the x_i-system in that system
v_i	Components of velocity of a particle in the x_i-system as seen by observer in that system
ν_i	Components of the unit exterior normal
$\psi_1, \psi_2, \psi_3, \phi$	Auxiliary functions
L	Undisturbed liquid height in the tank, from the liquid's c.g.
L_1	Distance from the c.g. of the liquid to an arbitrary point along the axis of symmetry
α_i	Effective accelerations = $-F_i + a_i$

NOMENCLATURE (Contd)

η	Free surface wave height
ϕ_{mn}	Eigenfunctions for the free vibrations of a system
ω_{mn}^2	Eigenfrequencies for the free vibrations of a system
K_{mn}	Nondimensional parameter $= \frac{L\omega_{mn}^2}{\alpha_3}$
ξ_{mn}	Time dependent portion of the surface wave height
$a_{mn}, b_{mn}, d_{mn}, e_{mn}, f_{mn}, h_{mn}, \gamma_{mn}$	Constants defined by (2.56)
V, S, FS	Actual liquid volume, surface, free surface
UV, US, UFS	Undisturbed liquid volume, surface, free surface
F_i'	Components of the hydrodynamic forces with center of rotation not at the c.g. of the liquid
$F_i^\star$	Components of the hydrodynamic forces in the x_i-system
T_i	Components of the hydrodynamic moments in the x_i-system
T_i'	Components of the hydrodynamic moments with center of rotation not at the c.g. of the liquid
I_{ij}	Constants defined by (2.61)
I_{ij}'	Constants defined by (3.39)
r, θ, z	Cylindrical polar coordinates
M	Mass of the liquid
β	Angle between the tangent line and the horizontal

SECTION 1

INTRODUCTION

The effect of liquid propellant motions must be considered in the design of most liquid-rocket-powered missiles and space vehicles. For the most part, the propellant motion problem is one of missile stability and control. Generally, the propellant motion interacts with both the control system dynamics and the vehicle dynamics, which also couple with each other. The natural frequencies of the oscillating propellants are often closer to the rigid body control frequencies of the vehicle than to the elastic body frequencies. Indeed, if the natural frequencies of the propellants in the tanks become too close to the control frequency of the vehicle or the natural frequency of the control sensor, the situation may become critical. Under these circumstances, the oscillating propellants exert large forces and moments, which, in turn, may saturate the control system and ultimately lead to structural failure of the vehicle. Thus, the response of the forces and moments exerted by the oscillating propellants on the vehicle must be sufficiently well defined analytically that the effects can be integrated into analyses of the overall system dynamic behavior. Generally, this is accomplished by a synthesis of the appropriate hydrodynamic equations, in which equivalent mechanical (mathematical) models composed of sets of simple spring-mass-dashpot or pendulums are devised. These are then combined with similar representations for other dynamic elements of the vehicle, and thus the overall system dynamic behavior can be determined by analog or digital techniques.

For the most part, recent papers in the field deal with specific details regarding the dynamic behavior of liquids in moving tanks under diverse conditions. The methods used in the majority of these studies are varied, and the assumptions are based on certain approximations which are often rather confusing and difficult to justify. It is beyond the purpose of this short review to discuss all these papers in detail. However, a study of these past efforts discloses a lack of agreement as to the exact analytic statement of the problem. In many instances the boundary conditions are in error, especially at the free liquid surface when the tank is undergoing pitching excitation. Generally, these errors can be attributed to conceptual misunderstandings, some of which, from one point of view fortunately, have not been particularly significant when the tank has been considered as rigid, or when the fundamental equations have been linearized.

Methods for calculating the dynamic response of liquids in moving tanks have been developed for cylinders of circular, elliptical, and rectangular cross sections with flat rigid bulkheads. These solutions make use of the technique of separation of variables. In fact, since the pertinent differential equation for the liquid behavior is Laplace's equation, a linearized solution can be found for any rigid cylindrical tank whose cross section is such that Laplace's equation is separable in the three dimensional cylindrical coordinate system, one of whose coordinate surfaces is the tank crosssectional boundary. Nevertheless, new and unusual tank configurations warrant consideration in the design of present day missiles and aerospace vehicles.

The object of this report is to rigorously derive the pertinent hydrodynamic equations for a missile tank of arbitrary shape which is allowed six degrees of freedom -- three rotational and three translational. Since the registering and control instruments used on missiles and aerospace vehicles are generally mounted directly on the vehicles, it is clear that measurements recorded on these are referred to vehicle-fixed or liquid tank-fixed axes, which are thus axes moving with respect to some inertial frame. It then becomes particuarly convenient to write all quantities, both absolute and relative, in terms of the moving coordinate system. It is in connection with the form of the equations of motion and the boundary conditions referred to tank-fixed axes that apparent conceptual misunderstandings have arisen in the literature.

SECTION 2

DERIVATION OF EQUATIONS FOR SLOSHING MOTIONS IN TANKS OF ARBITRARY SHAPE

2.1 BASIC EQUATIONS FOR SIX DEGREES OF FREEDOM. Consider a mobile tank of arbitrary geometry, partially filled with a perfect incompressible liquid. Suppose the tank to be subjected to a constant or nearly constant acceleration along a given direction. Then, in the absence of all other accelerations, the free surface of the liquid in the tank becomes a plane normal to the direction of constant acceleration; i.e., this axis is colinear with the exterior normal to the undisturbed free surface of the liquid in the tank. Assume the tank to undergo angular and linear accelerations in three mutually orthogonal directions, one direction being the direction of constant acceleration. These disturbances are presumed to be small; the squares and products of these quantities and their derivatives are small in comparison with the quantities themselves, hence they will be neglected (i.e., only linear effects are considered).

It is convenient to refer the motion of the liquid to a translating and rotating coordinate system fixed in the tank. In the ensuing analysis, the convention used is that Latin indices take on the values 1, 2, 3. If in some expression an index occurs twice, the expression is to be summed with respect to that index over its range of values.

Let y_i be the coordinates of a point referred to a Cartesian coordinate system, y_i, fixed in space, and let x_i be the coordinates of a translating and rotating system, x_i, (see Figure 1). The origin of the system x_i is assumed to be located at the undisturbed center of gravity of the liquid. The coordinates of the two systems are related by

$$y_i = z_i + a_{ji} x_j \tag{2.1}$$

$$x_i = a_{ij} (y_j - z_j) \tag{2.2}$$

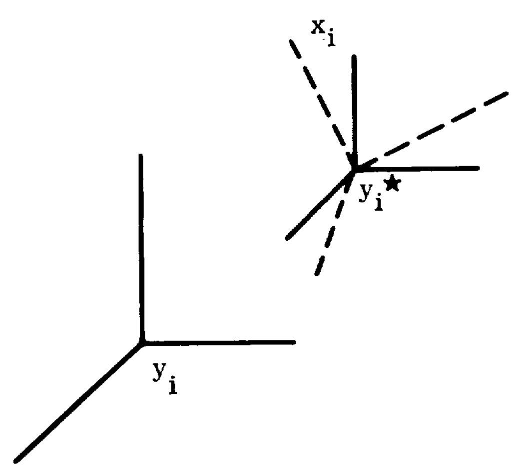

Figure 1. Coordinate Systems

If $\bar{\vartheta}_i$ are components of a vector referred to y_i, the components, ϑ_i, of this vector in system x_i are related to $\bar{\vartheta}_i$ by

$$\vartheta_i = a_{ij} \bar{\vartheta}_j \tag{2.3}$$

$$\bar{\vartheta}_i = a_{ji} \vartheta_j \tag{2.4}$$

In (2.1) through (2.4), z_i (components measured in the fixed system) measures the instantaneous displacement of the origin of x_i with respect to the origin of y_i, and $a_{ij} = \cos(x_i, y_j)$ measures the instantaneous rotation of the x_i axis with respect to y_i. The a_{ij} are functions of time satisfying the equations

$$a_{ik}\, a_{jk} = \delta_{ij} \tag{2.5}$$

where δ_{ij} is the Kronecker delta; i.e.

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$$

and

$$a_{ik} \frac{da_{jk}}{dt} = -\epsilon_{ijk}\, \omega_k \qquad \text{(see Appendix A)} \tag{2.6}$$

where ϵ_{ijk} is the alternating tensor; i.e.

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if ijk is a cyclic permutation of } 1,2,3 \\ -1 \text{ if ijk is an acyclic permutation of } 1,2,3 \\ 0 \text{ if any two subscripts are equal} \end{cases}$$

The ω_j are the components of the angular velocity as measured in the rotating system.

Differentiate (2.1) with respect to t to get

$$\bar{q}_k \equiv \frac{dy_i}{dt} = \frac{dz_i}{dt} + \left[\frac{d}{dt} a_{ji}\right] x_j + a_{ji} \frac{dx_j}{dt} \tag{2.7}$$

The components of $\frac{dy_i}{dt}$ referred to the moving system are

$$q_k \equiv a_{ki} \frac{dy_i}{dt} = a_{ki} \frac{dz_i}{dt} + \left[a_{ki} \frac{da_{ji}}{dt}\right] x_j + a_{ki}\, a_{ji} \frac{dx_j}{dt} \tag{2.8}$$

Use (2.5), (2.6), and the properties of the alternating tensor to reduce (2.8) to

$$q_k = u_k + \epsilon_{kij}\, \omega_i\, x_j + \frac{d}{dt} x_k \tag{2.9}$$

where $u_k \equiv a_{ki} \frac{dz_i}{dt}$ are the components of the translational velocity of the origin of x_i referred to the system x_i.

Referred to a fixed axis, the Eulerian equations of motion for an incompressible liquid are

$$\frac{d}{dt}\,\bar{q}_i = \bar{F}_i - \frac{1}{\rho}\frac{\partial p}{\partial y_i} \tag{2.10}$$

where $\bar{F}_i$ are the components of the body force vector, p is the pressure, ρ is the density of the liquid, and $\frac{d}{dt}$ represents the total (material) derivative.

From (2.7) and (2.8)

$$q_i = a_{ik}\bar{q}_k \quad \text{and} \quad \bar{q}_k = a_{ik}\, q_i \tag{2.11}$$

Differentiate the first equation in (2.11) with respect to t.

$$\frac{dq_i}{dt} = \left[\frac{d}{dt}\, a_{ik}\right] \bar{q}_k + a_{ik}\frac{d}{dt}\,\bar{q}_k \tag{2.12}$$

Then use (2.6), (2.10), and (2.11) to obtain

$$\frac{dq_i}{dt} = -\epsilon_{jik}\,\omega_k\, q_j + a_{ik}\left[\bar{F}_k - \frac{1}{\rho}\frac{\partial p}{\partial y_k}\right] \tag{2.13}$$

or

$$\frac{dq_i}{dt} + \epsilon_{ikj}\,\omega_k\, q_j = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{2.14}$$

In (2.14) $F_i = a_{ik}\,\bar{F}_k$ are the components of the body force measured in the moving system, x_i, and

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial y_k}\,\frac{\partial y_k}{\partial x_i} = \frac{\partial p}{\partial y_k}\left[\frac{\partial z_k}{\partial x_i} + a_{jk}\,\delta_{ij}\right] = a_{ik}\frac{\partial p}{\partial y_k} \tag{2.15}$$

Substitute the value of q_k from (2.9) into (2.14) to obtain

$$\frac{du_i}{dt} + \epsilon_{ikj}\,\omega_k\, u_j + 2\epsilon_{ikj}\,\omega_k\frac{dx_j}{dt} + \epsilon_{ikj}\frac{d\omega_k}{dt}\, x_j$$

$$+ \epsilon_{ikj}\,\omega_k\,\epsilon_{j\ell m}\,\omega_\ell\, x_m + \frac{d^2}{dt^2}\, x_i = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{2.16}$$

Let $a_i = \frac{du_i}{dt} + \epsilon_{ikj}\, \omega_k\, u_j$ and $\frac{dx_i}{dt} = v_i$, where the a_i are the components of the absolute acceleration of the origin of the translating and rotating system, x_i, in that system. With this notation, (2.16) becomes

$$\frac{d}{dt} v_i + 2\epsilon_{ikj}\, \omega_k\, v_j = F_i - a_i - \epsilon_{ikj} \frac{d\omega_k}{dt} x_j$$

$$- \epsilon_{ikj}\, \epsilon_{j\ell m}\, \omega_k\, \omega_\ell\, x_m - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \tag{2.17}$$

(Recall that $\epsilon_{ikm}\, \epsilon_{psm} = \delta_{ip}\, \delta_{ks} - \delta_{is}\, \delta_{kp}$) (2.18)

Referred to the moving axis, the equation of continuity for the Eulerian viewpoint is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \left[\rho v_i\right] = 0 \qquad \text{(see Reference 1, p. 12)} \tag{2.19}$$

For an incompressible liquid

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} = 0 \tag{2.20}$$

Thus it follows from the continuity equation that

$$\frac{\partial}{\partial x_i} v_i = 0 \tag{2.21}$$

If the liquid in the tank of the missile behaves as an irrotational liquid

$$\frac{\partial v_k}{\partial x_i} = \frac{\partial v_i}{\partial x_k} \tag{2.22}$$

Differentiate (2.17) with respect to x_i and assume that the orders of differentiation may be interchanged to obtain

$$\left[\frac{\partial}{\partial t} \frac{\partial v_i}{\partial x_i} + v_j \frac{\partial^2 v_i}{\partial x_j \partial x_i}\right] + 2\epsilon_{ikj}\, \omega_k \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}(F_i - a_i) - \epsilon_{ikj} \frac{d\omega_k}{dt} \delta_{ij}$$

$$- \epsilon_{ikj}\, \epsilon_{jnm}\, \omega_k\, \omega_n\, \delta_{im} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} - \frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} \tag{2.23}$$

The first term on the left hand side of (2.23) vanishes because of the incompressibility condition. The second term vanishes because of irrotational motion

$$(\epsilon_{ikj}\,\omega_k \frac{\partial v_j}{\partial x_i} = \epsilon_{jki}\,\omega_k \frac{\partial v_i}{\partial x_j} \qquad \text{(interchange dummy indices)}$$

$$= \epsilon_{jki}\,\omega_k \frac{\partial v_j}{\partial x_i} \qquad \text{(from 2.22)}$$

$$= -\epsilon_{ikj}\,\omega_k \frac{\partial v_j}{\partial x_i}$$

and a quantity equal to its negative must equal zero.)

The first term on the right hand side of (2.23) vanishes because $F_i - a_i$ is independent of x_i; the second term vanishes because $\epsilon_{iki} = 0$. From (2.18)

$$-\epsilon_{ikj}\,\epsilon_{jni} = -\epsilon_{kji}\,\epsilon_{jni}$$

$$= -(\delta_{kj}\,\delta_{jn} - \delta_{kn}\,\delta_{jj})$$

$$= -(\delta_{kn} - 3\,\delta_{kn}) = 2\,\delta_{kn}$$

Thus (2.23) reduces to

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} = 2\,\omega_i\,\omega_i - \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} \tag{2.24}$$

Linearize (2.24) to obtain

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i\, \partial x_i} = 0 \qquad \text{(in the volume of the liquid)} \tag{2.25}$$

which must be solved subject to boundary conditions at the free surface and the tank walls.

The boundary conditions may be obtained from the principle of continuity which states that the liquid and boundary surface with which contact is maintained have equal components of velocity normal to the surface. Let $x_i^\star$ be the coordinates of a surface measured relative to the tank-fixed coordinate system, x_i. Then the boundary conditions may be expressed on all surfaces as

$$\nu_i\, v_i = \nu_i\, \dot{x}_i^\star \tag{2.26}$$

(In the remaining analysis a total time derivative in the moving system is designated by a dot.)

To use (2.17), linearize the equation to obtain

$$\dot{v}_i = F_i - a_i - \epsilon_{ikj}\,\dot{\omega}_k\,x_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{2.27}$$

Since the normal vector is assumed to be independent of time, differentiate (2.26) with respect to t and substitute the value of $\dot{v}_i$ from (2.27) into the resulting expression to obtain

$$\nu_i\left[F_i - a_i - \epsilon_{ikj}\,\dot{\omega}_k\,x_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i}\right] = \nu_i\,\ddot{x}_i^{\star} \tag{2.28}$$

Thus, the boundary conditions are

$$\nu_i\,\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \nu_i\left[F_i - a_i - \epsilon_{ikj}\,\dot{\omega}_k\,x_j - \ddot{x}_i^{\star}\right] \quad \text{(over all surfaces)} \tag{2.29}$$

2.2 FORMULATION OF BOUNDARY VALUE PROBLEMS

2.2.1 Definitions. The problem at this point is to solve the partial differential equation

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i\,\partial x_i} = 0 \quad \text{(throughout the volume of the liquid)} \tag{2.30}$$

subject to (2.29). For convenience, let $\phi(x) \sim \phi(x_1, x_2, x_3)$ etc. in the following discussion. To simplify this problem, let

$$\frac{p}{\rho} = -\alpha_i\,x_i + L^2\,\psi_i\,\dot{\omega}_i - \left[\dot{\omega}_1\,x_2\,x_3 + \dot{\omega}_2\,x_1\,x_3 + \dot{\omega}_3\,x_1\,x_2\right] + \phi + C(t) \tag{2.31}$$

where

$$\psi_i = \psi_i(x)$$

$$\phi = \phi(x, t)$$

$$\alpha_i = -F_i + a_i$$

α_3 is assumed to be constant (constant thrust) and L is the distance between the undisturbed center of gravity and the undisturbed free surface of the fluid, measured along the x_3-axis

$$\frac{\partial^2 \psi_i}{\partial x_j \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_j} = 0 \tag{2.32}$$

The boundary conditions for ψ_i are defined over all surfaces as

$$\nu_i \frac{\partial \psi_1}{\partial x_i} = \frac{2 x_3}{L^2} \nu_2 \tag{2.33}$$

$$\nu_i \frac{\partial \psi_2}{\partial x_i} = \frac{2 x_1}{L^2} \nu_3 \tag{2.34}$$

$$\nu_i \frac{\partial \psi_3}{\partial x_i} = \frac{2 x_2}{L^2} \nu_1 \tag{2.35}$$

The form of (2.31) and conditions (2.33), (2.34), and (2.35) are chosen so as to make the boundary conditions (2.29) reduce to the simple form of (2.38). Substitute the value of $\frac{p}{\rho}$ from (2.31) into (2.29) and make use of the boundary conditions on the ψ_i to obtain

$$\nu_i \frac{\partial \phi}{\partial x_i} = -\nu_i \ddot{x}_i^\star \qquad \text{(over all surfaces)} \tag{2.36}$$

Thus

$$\nu_i \frac{\partial \phi}{\partial x_i} = 0 \qquad \text{(along the rigid tank surfaces)} \tag{2.37}$$

and

$$\nu_i \frac{\partial \phi}{\partial x_i} = -\nu_i \ddot{x}_i^\star \qquad \text{(at the free surface of the liquid)} \tag{2.38}$$

Assume that the free surface of the liquid is given by

$$x_i^\star = \left[L + \eta(x_1, x_2, t)\right] \delta_{i3} \tag{2.39}$$

η is assumed to be small (small shallow oscillations) so that the normal to the middle surface is approximated by the normal to the undisturbed free surface; i.e.

$$\nu_i \approx \delta_{i3} \tag{2.40}$$

on the free surface of the liquid, and the value of $\frac{\partial \phi}{\partial x_3}$ on the free surface is approximately equal to its value at $x_3 = L$. Thus, (2.38) becomes

$$\left(\frac{\partial \phi}{\partial x_3}\right)_{x_3 = L} = -\ddot{\eta} \tag{2.41}$$

2.2.2 Forces and Moments. The dynamic condition used to determine η is that the pressure be independent of position on the free surface. On the free surface the pressure is given by

$$\left(\frac{p}{\rho}\right)_{x_3 = L+\eta} = C(t) - \alpha_3\, L + \Big[-\alpha_1\, x_1 - \alpha_2\, x_2 - \alpha_3\, \eta + L^2\, \psi_i\, \dot{\omega}_i$$

$$- \dot{\omega}_1\, x_2\, (L+\eta) - \dot{\omega}_2\, x_1\, (L+\eta) - \dot{\omega}_3\, x_1\, x_2 + \phi\Big] \tag{2.42}$$

Thus, the dynamic condition requires that the bracketed term in (2.42) must vanish. Assume that

$$(\psi_i)_{x_3 = L+\eta} \approx (\psi_i)_{x_3 = L}$$

$$(\phi)_{x_3 = L+\eta} \approx (\phi)_{x_3 = L}$$

and that ω_i varies slowly in time; thus products of $\dot{\omega}_i$ with itself and with η are sufficiently small in comparison with the terms themselves that these products may be neglected. Thus

$$\alpha_3 \eta = -\alpha_1\, x_1 - \alpha_2\, x_2 + L^2\, \psi_i\, \dot{\omega}_i - \dot{\omega}_1\, x_2\, L - \dot{\omega}_2\, x_1\, L - \dot{\omega}_3\, x_1\, x_2 + \phi \tag{2.43}$$

where ψ_i and ϕ are evaluated at $x_3 = L$

Consider the free vibrations of a system, described by

$$\left.\begin{array}{lll} \dfrac{\partial^2 \phi_o}{\partial x_i\, \partial x_i} = 0 & & \text{(throughout the volume of the liquid)} \\[2ex] \nu_i\, \dfrac{\partial \phi_o}{\partial x_i} = 0 & & \text{(at the tank walls)} \\[2ex] \dfrac{\partial^2 \phi_o}{\partial t^2} + \alpha_3\, \dfrac{\partial \phi_o}{\partial x_3} = 0 & & \text{(at the undisturbed free surface of the liquid, } x_3 = L) \end{array}\right\} \tag{2.44}$$

Assume $\phi_o = \bar{\phi}_o\, (x_1\, x_2\, x_3)\, e^{i\omega t}$ to obtain

$$
\left.\begin{aligned}
&\frac{\partial^2 \bar{\phi}_o}{\partial x_i \, \partial x_i} = 0 && \text{(throughout the volume of the liquid)} \\
&\nu_i \frac{\partial \bar{\phi}_o}{\partial x_i} = 0 && \text{(at the tank walls)} \\
&\frac{\partial \bar{\phi}_o}{\partial x_3} = \frac{\omega^2}{\alpha_3} \bar{\phi}_o && \text{(at } x_3 = L\text{)}
\end{aligned}\right\} \tag{2.45}
$$

Let the eigenfunctions be given by $\phi_{mn}(x_1, x_2, x_3)$ and the corresponding values of ω^2 by ω^2_{mn}. Define K_{mn} by

$$
\frac{\omega^2_{mn}}{\alpha_3} \equiv \frac{K_{mn}}{L} \tag{2.46}
$$

Then the ϕ_{mn} have the properties

$$
\left.\begin{aligned}
&\frac{\partial^2 \phi_{mn}}{\partial x_i \, \partial x_i} = 0 && \text{(throughout the volume of liquid)} \\
&\nu_i \frac{\partial \phi_{mn}}{\partial x_i} = 0 && \text{(at the tank walls)} \\
&\frac{\partial \phi_{mn}}{\partial x_3} = \frac{K_{mn}}{L} \phi_{mn} && \text{(at } x_3 = L\text{)}
\end{aligned}\right\} \tag{2.47}
$$

and

$$
\phi_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \, \phi_{mn} \, e^{i\omega_{mn} t}
$$

Assume that the eigenfrequencies and eigenfunctions associated with this boundary value problem have been determined. Express ϕ and η in terms of the eigenfunctions for the free vibration; i.e., assume

$$
\phi(x_1, x_2, x_3, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn}(t) \, \phi_{mn}(x_1, x_2, x_3) \tag{2.48}
$$

and

$$
\eta(x_1, x_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn}(t) \, \phi_{mn}(x_1, x_2, L) \tag{2.49}
$$

The relationships between the $\lambda_{mn}(t)$ and $\xi_{mn}(t)$ are determined from (2.41), using the properties of $\phi_{mn}(x_1, x_2, x_3)$ in (2.47) as follows.

$$\left.\begin{aligned}
\left(\frac{\partial \phi}{\partial x_3}\right)_{x_3=L} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn}(t) \frac{\partial \phi_{mn}}{\partial x_3}(x_1, x_2, L) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn}(t) \frac{K_{mn}}{L} \phi_{mn}(x_1, x_2, L) \\
&= -\ddot{\eta} \\
&= -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\xi}_{mn}(t)\, \phi_{mn}(x_1, x_2, L)
\end{aligned}\right\} \qquad (2.50)$$

Since this relationship must hold for all x_1, x_2, and t

$$\lambda_{mn}(t) = -\frac{L}{K_{mn}} \dot{\xi}_{mn}(t) \qquad (2.51)$$

and

$$\phi(x_1, x_2, x_3, t) = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{L}{K_{mn}} \dot{\xi}_{mn}\, \phi_{mn}(x_1, x_2, x_3) \qquad (2.52)$$

Thus, from (2.31), the pressure is given as

$$\frac{p}{\rho} = C(t) - \alpha_i x_i + L^2 \psi_i \dot{\omega}_i - \left[\dot{\omega}_1 x_2 x_3 + \dot{\omega}_2 x_1 x_3 + \dot{\omega}_3 x_1 x_2\right]$$

$$- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{L}{K_{mn}} \ddot{\xi}_{mn}\, \phi_{mn}(x_1, x_2, x_3) \qquad (2.53)$$

With this formulation, boundary condition (2.43) becomes

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{L}{K_{mn}} \ddot{\xi}_{mn} + \alpha_3 \xi_{mn}\right] \phi_{mn}(x_1, x_2, L) = -\alpha_1 x_1 - \alpha_2 x_2$$

$$-(x_2 L - L^2 \psi_1)\, \dot{\omega}_1 - (x_1 L - L^2 \psi_2)\, \dot{\omega}_2 - (x_1 x_2 - L^2 \psi_3)\, \dot{\omega}_3 \qquad (2.54)$$

where the ψ_i on the right side of the equation are evaluated at $x_3 = L$. Assume that the right side of (2.54) may be expanded in terms of $\phi_{ij}(x_1, x_2, L)$. This expansion will be valid at least in the interior of the region. By making use of the orthogonality properties of the ϕ_{ij}, (2.54) can be rewritten in the form (see Appendix B for details)

$$\dot{\xi}_{mn} + \alpha_3 \frac{K_{mn}}{L} \xi_{mn} = -K_{mn}\, a_{mn}\, \alpha_1 - K_{mn}\, b_{mn}\, \alpha_2$$
$$-LK_{mn}\left[b_{mn} - h_{mn}\right]\dot{\omega}_1 - LK_{mn}\left[a_{mn} - d_{mn}\right]\dot{\omega}_2$$
$$-LK_{mn}\left[e_{mn} - f_{mn}\right]\dot{\omega}_3 \qquad (2.55)$$

where

$$a_{mn}\, V\, \gamma_{mn} = \int_{UFS} x_1\, \phi_{mn}\, dS$$

$$b_{mn}\, V\gamma_{mn} = \int_{UFS} x_2\, \phi_{mn}\, dS$$

$$h_{mn}\, VK_{mn}\, \gamma_{mn} = 2\int_{US} x_3\, \phi_{mn}\, \nu_2\, dS$$

$$d_{mn}\, VK_{mn}\, \gamma_{mn} = 2\int_{US} x_1\, \phi_{mn}\, \nu_3\, dS$$

$$e_{mn}\, V\, L\, \gamma_{mn} = \int_{UFS} x_1\, x_2\, \phi_{mn}\, dS \qquad (2.56)$$

$$f_{mn}\, VK_{mn}\, \gamma_{mn} = 2\int_{US} x_2\, \phi_{mn}\, \nu_1\, dS$$

$$\gamma_{mn}\, V = L\int_{UFS} (\phi_{mn})^2\, dS$$

$$V = \int_{UV} dV$$

The forces on the liquid-tank system have the components, referred to the x_i-axis

$$F_i^{\star} = \int_S p\,\nu_i\,dS = \int_S p\,\nu_j\,\delta_{ij}\,dS = \int_S p\,\nu_j \frac{\partial x_i}{\partial x_j}\,dS$$

$$= \int_S x_i\,\nu_j \frac{\partial p}{\partial x_j}\,dS \qquad \text{(Theorem III)}$$

$$= \int_V \frac{\partial p}{\partial x_i}\,dV \qquad \text{(Theorem I)} \tag{2.57}$$

In the equations above, the surface integrals are over the entire surface of the liquid (i.e., over the free surface and tank walls), and the volume integral is over the entire volume occupied by the liquid. Substitute the value of the pressure from (2.53) into (2.57) to obtain (see Appendix C for the details)

$$F_i^{\star} = -M\alpha_i - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} (a_{mn}\,\delta_{i1} + b_{mn}\,\delta_{i2})\,\ddot{\xi}_{mn} \tag{2.58}$$

where $M = \rho V$ is the mass of the liquid.

The moments may be calculated from

$$T_i = \int_S \epsilon_{ijk}\,x_j\,\nu_k\,p\,dS$$

$$= \int_V \epsilon_{ijk} \frac{\partial}{\partial x_k}(x_j p)\,dV \qquad \text{(Theorem I)}$$

$$= \int_V \epsilon_{ijk}\,x_j \frac{\partial p}{\partial x_k}\,dV \tag{2.59}$$

Substitute the value of $\frac{\partial p}{\partial x_k}$ from (2.53) into (2.59) to obtain (see Appendix D for the details)

$$T_i = -\dot{\omega}_1 \left[I_{11}\,\delta_{i1} - I_{21}\,\delta_{i2} - I_{31}\,\delta_{i3}\right] - \dot{\omega}_2 \left[-I_{12}\,\delta_{i1} + I_{22}\,\delta_{i2} - I_{32}\,\delta_{i3}\right]$$

$$- \dot{\omega}_3 \left[-I_{13}\,\delta_{i1} - I_{23}\,\delta_{i2} + I_{33}\,\delta_{i3}\right] - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} \Big[\Big\{ L(b_{mn}$$

$$- h_{mn}) \ddot{\xi}_{mn} + \alpha_3 b_{mn} \xi_{mn} \Big| \delta_{i1} + \Big| L(a_{mn} - d_{mn}) \ddot{\xi}_{mn}$$

$$- \alpha_3 a_{mn} \xi_{mn} \Big| \delta_{i2} + \Big| L(e_{mn} - f_{mn}) \ddot{\xi}_{mn} \Big| \delta_{i3} \Big| \qquad (2.60)$$

where

$$\left.\begin{aligned}
I_{11} &= \rho \int_{UV} (x_2^2 + x_3^2)\, dV - 4\rho \int_{UV} x_3^2\, dV + 2\rho L^2 \int_{US} x_3 \psi_1 \nu_2\, dS \\
I_{21} &= \rho \int_{UV} x_1 x_2\, dV - 2\rho L^2 \int_{US} x_1 \psi_1 \nu_3\, dS \\
I_{31} &= \rho \int_{UV} x_1 x_3\, dV - 2\rho L^2 \int_{US} x_2 \psi_1 \nu_1\, dS \\
I_{12} &= \rho \int_{UV} x_1 x_2\, dV - 2\rho L^2 \int_{US} x_3 \psi_2 \nu_2\, dS \\
I_{22} &= \rho \int_{UV} (x_1^2 + x_3^2)\, dV - 4\rho \int_{UV} x_1^2\, dV + 2\rho L^2 \int_{US} x_1 \psi_2 \nu_3\, dS \\
I_{32} &= \rho \int_{UV} x_2 x_3\, dV + 2\rho L^2 \int_{US} x_2 \psi_2 \nu_1\, dS \\
I_{13} &= \rho \int_{UV} x_1 x_3\, dV + 2\rho L^2 \int_{US} x_3 \psi_3 \nu_2\, dS \\
I_{23} &= \rho \int_{UV} x_2 x_3\, dV + 2\rho L^2 \int_{US} x_1 \psi_3 \nu_3\, dS \\
I_{33} &= \rho \int_{UV} (x_1^2 + x_2^2)\, dV - 4\rho \int_{UV} x_2^2\, dV + 2\rho L^2 \int_{US} x_2 \psi_3 \nu_1\, dS
\end{aligned}\right\} \qquad (2.61)$$

and the other constants are given by (2.56).

2.2.3 Results. For a mobile tank with arbitrary geometry, subject to acceleration and having six degrees of freedom (three translational and three rotational), the pertinent quantities have been obtained. That is,

a. The moments are given by (2.60).

b. The forces are given by (2.58).

c. The pressure is given by (2.53).

d. The surface wave height is given by (2.49).

The eigenfunctions in these equations are given by the solutions of the following boundary value problems.

$$\frac{\partial^2 \phi_{mn}}{\partial x_i \partial x_i} = \frac{\partial^2 \psi_k}{\partial x_i \partial x_i} = 0 \qquad \text{(through the volume of the liquid)} \tag{2.62}$$

$$\nu_i \frac{\partial \phi_{mn}}{\partial x_i} = 0 \qquad \text{(at the tank boundary)} \tag{2.63}$$

$$\frac{\partial \phi_{mn}}{\partial x_3} = \frac{K_{mn}}{L} \phi_{mn} \qquad \text{(on the quiescent free surface)} \tag{2.64}$$

$$\left.\begin{array}{ll} L^2 \nu_i \dfrac{\partial \psi_1}{\partial x_i} = 2 x_3 \nu_2 & \text{(over all surfaces)} \\ L^2 \nu_i \dfrac{\partial \psi_2}{\partial x_i} = 2 x_1 \nu_3 & \text{(over all surfaces)} \\ L^2 \nu_i \dfrac{\partial \psi_3}{\partial x_i} = 2 x_2 \nu_1 & \text{(over all surfaces)} \end{array}\right\} \tag{2.65}$$

These boundary value problems cannot be solved until a specific tank shape is given. For the special case where the tank has an axis of symmetry and three degrees of freedom, it is possible to develop a technique to obtain the eigenfunctions and eigenvalues numerically. This is done in Reference 2.

The ξ_{mn} are given by (2.55), where the K_{mn} are given by the solution to the above boundary value problem.

SECTION 3

DEGENERATE CASE (THREE DEGREES OF FREEDOM)

3.1 REDUCTION OF THE EQUATIONS FROM SECTION 2. If the problem is described by one translational component, one rotational component, and the constant acceleration along the axis of the missile, the computations involved in finding the pertinent quantities will simplify as shown in the following paragraph.

In the equations of the previous sections, let $\omega_2 = \omega_3 = u_1 = 0$.

The moments (from 2.60) become

$$T_i = - \dot{\omega}_1 \left[I_{11}\,\delta_{i1} - I_{21}\,\delta_{i2} - I_{31}\,\delta_{i3} \right] - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{nm} \Big[\Big\{ L\,(b_{mn} - h_{mn})\, \ddot{\xi}_{mn}$$

$$+ \alpha_3\, b_{mn}\, \xi_{mn} \Big\} \delta_{i1} + \Big\{ L\,(a_{mn} - d_{mn})\, \ddot{\xi}_{mn} - \alpha_3\, a_{mn}\, \xi_{mn} \Big\} \delta_{i2}$$

$$+ \Big\{ L\,(e_{mn} - f_{mn})\, \ddot{\xi}_{mn} \Big\} \delta_{i3} \Big] \tag{3.1}$$

The forces (from (2.58)) are

$$F_i^{\star} = - \alpha_3\, M\, \delta_{i3} - M\, \alpha_2\, \delta_{i2} - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn}\,(a_{mn}\,\delta_{i1} + b_{mn}\,\delta_{i2})\, \ddot{\xi}_{mn} \tag{3.2}$$

The pressure (from (2.53)) is

$$p = \rho \Big[C(t) - \alpha_2\, x_2 - \alpha_3\, x_3 - (x_2\, x_3 - L^2\, \psi_1)\, \dot{\omega}_1$$

$$- L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\phi_{mn}}{K_{mn}}\, \ddot{\xi}_{mn} \Big] \tag{3.3}$$

The surface wave height (from (2.49)) is given by

$$\eta\,(x_1, x_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn}(t)\ \phi_{mn}\,(x_1, x_2, L) \tag{3.4}$$

where the ϕ_{mn} is given by the solution to the boundary value problem described by (2.62), (2.63), and (2.64) and the ξ_{mn} are given from

$$\dot{\xi}_{mn} + \alpha_3 \frac{K_{mn}}{L} \xi_{mn} = - K_{mn} b_{mn} \alpha_2 - L K_{mn} \left[b_{mn} - h_{mn} \right] \dot{\omega}_1 \tag{3.5}$$

3.2 <u>RESULTS FOR AN AXIS OF SYMMETRY</u>. In most cases of interest, especially in the design of missiles and launch vehicles, the tank is symmetric or nearly symmetric about the axis of constant acceleration, the x_3-axis. For this situation the equations for three degrees of freedom are further simplified. Refer the tank to cylindrical polar coordinates defined by

$$\left.\begin{aligned} x_1 &= r \cos \theta \\ x_2 &= r \sin \theta \\ x_3 &= z \end{aligned}\right\} \tag{3.6}$$

Since the tank is symmetric about the x_3-axis, the undisturbed boundary surface of the liquid enclosed by the tank is a surface of revolution formed by revolving a curve, shown as ABC in Figure 2, about the x_3-axis.

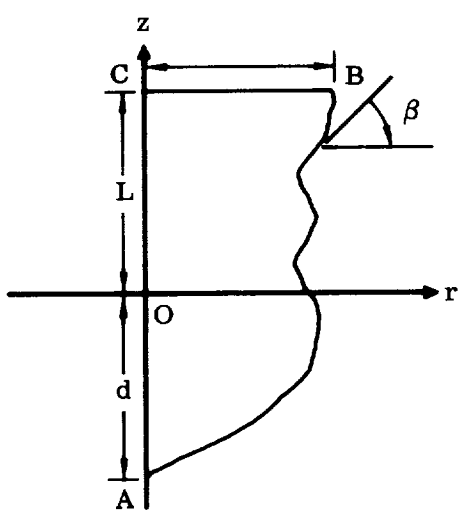

Figure 2. Arbitrary Tank Cross Section

The origin O is located at the undisturbed center of gravity of the liquid. The curve A-B is formed by the tank profile, and the curve B-C is formed by the intersection of the quiescent free surface and a plane parallel to and including the x_3-axis. The line AOC is given by the portion of the line $r = 0$, which is interior to the liquid volume. In some cases (e.g., a tank formed with concentric cylinders), points A and C may coincide. Then, C does not necessarily lie on the x_3-axis.

For cylindrical polar coordinates, the components of the exterior normal are given by

$$\left.\begin{aligned} \nu_1 &= \sin \beta \cos \theta \\ \nu_2 &= \sin \beta \sin \theta \\ \nu_3 &= -\cos \beta \end{aligned}\right\} \tag{3.7}$$

where β is defined by

$$\tan\beta = \frac{dz}{dr} \tag{3.8}$$

on the curve ABC.

The element of arc length is given by

$$ds^2 = dz^2 \csc^2\beta = dr^2 \sec^2\beta \tag{3.9}$$

To satisfy the boundary conditions and reduce the boundary value problems to ones independent of θ, let

$$\psi_1 = \sin\theta \, \Psi_1 (r, z), \quad \phi_{mn} = \sin\theta \, \Phi_n (r, z)$$

The boundary value problems become

$$\frac{\partial^2 \Psi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi_1}{\partial r} - \frac{1}{r^2} \Psi_1 + \frac{\partial^2 \Psi_1}{\partial z^2} = 0 \qquad \text{(interior to ABCOA)} \tag{3.10}$$

$$\sin\beta \frac{\partial \Psi_1}{\partial r} - \cos\beta \frac{\partial \Psi_1}{\partial z} = \frac{2z}{L^2} \sin\beta \qquad \text{(on ABC)} \tag{3.11}$$

and

$$\frac{\partial^2 \Phi_n}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_n}{\partial r} - \frac{1}{r^2} \Phi_n + \frac{\partial^2 \Phi_n}{\partial z^2} = 0 \qquad \text{(interior to ABCOA)} \tag{3.12}$$

$$\frac{\partial \Phi_n}{\partial z} = \frac{K_n}{L} \Phi_n \qquad \text{(along BC)} \tag{3.13}$$

$$\sin\beta \frac{\partial \Phi_n}{\partial r} - \cos\beta \frac{\partial \Phi_n}{\partial z} = 0 \qquad \text{(along AB)} \tag{3.14}$$

With this formulation the forces and moments reduce to

$$T_i = - \dot{\omega}_1 I_{11} \delta_{i1} - M \sum_{n=1}^{\infty} \gamma_n \left| L(b_n - h_n) \ddot{\xi}_n + \alpha_3 b_n \xi_n \right| \delta_{i1} \tag{3.15}$$

$$F_i^{\star} = -\alpha_3 M \delta_{i3} - M \alpha_2 \delta_{i2} - M \sum_{n=1}^{\infty} \gamma_n b_n \dot{\xi}_n \delta_{i2} \tag{3.16}$$

where

$$\left.\begin{aligned} V \gamma_n b_n &= \pi \int_C^B r^2 \Phi_n (r, L) \, dr \\ K_n V \gamma_n h_n &= 2\pi \int_A^B zr \, \Phi_n (r, z) \, dz \\ V \gamma_n &= \pi L \int_C^B r \left[\Phi_n (r, L)\right]^2 dr \end{aligned}\right\} \tag{3.17}$$

$$I_{11} = \rho \int_{UV} (r^2 \cos^2\theta + z^2) \, dV - 4\rho \int_{UV} z^2 \, dV + 2\rho L^2 \pi \int_A^B zr \Psi_1 \, dz$$

The pressure and surface wave height are given by (3.3) and (3.4).

A digital computer program has been developed to find the coefficients in (3.17) along with the eigenvalues and eigenfunctions for any tank whose cross section is composed of portions of ellipses, circles, parabolas, and/or straight lines. The method will handle any ring-shaped tank or baffles which are adjacent to a rigid wall. The procedure and program are described in Report GD|A-DDE64-062 (see Reference 2).

3.3 CENTER OF MASS NOT COINCIDENT WITH THE CENTER OF ROTATION

3.3.1 Analytic Hydrodynamic Formulas. The motion of the liquid in a missile tank will obviously interact with the control system and vehicle dynamics. In order to incorporate the liquid effects into an analysis of the entire missile behavior for control purposes, the pertinent quantities must be referred to a coordinate system, fixed in the missile, which is no longer located at the center of gravity of the liquid.

For a three-degree-of-freedom analysis, the origin will be translated along the x_3-axis to a point, L_1, below its original position. Call this new coordinate system x_i'. The problem is described by (from (2.30) and (2.29))

$$\frac{1}{\rho} \frac{\partial^2 p'}{\partial x_i' \partial x_i'} = 0 \qquad \text{(throughout the volume of the liquid)} \tag{3.18}$$

$$\frac{1}{\rho} \nu_i' \frac{\partial p'}{\partial x_i'} = -\nu_i' \left(\alpha_i + \epsilon_{ijk} \dot{\omega}_j x_k' + \ddot{x}_i^{\star\prime}\right) \qquad \text{(on the boundary)} \tag{3.19}$$

The primed system is related to the unprimed system with origin at the center of gravity of the liquid by

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3 + L_1 \tag{3.20}$$

$$\therefore \; \frac{\partial}{\partial x_i'} = \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \nu_i' = \nu_i$$

Thus (3.18) and (3.19) become, for three degrees of freedom

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \, \partial x_i} = 0 \tag{3.21}$$

$$\frac{1}{\rho} \nu_i \frac{\partial p}{\partial x_i} = - \alpha_2 \nu_2 - \alpha_3 \nu_3 - \dot{\omega}_1 \left(x_2 \nu_3 - (x_3 + L_1) \nu_2\right) - \nu_1 \ddot{x}_i^{\star\prime} \tag{3.22}$$

Since the procedure used from this point on is exactly that used in Section 2, many of the details will be omitted.

Assume

$$\frac{p}{\rho} = - \alpha_2 \, x_2 - \alpha_3 \, x_3 - \omega_1 \left[(x_3 - L_1) \, x_2 - L^2 \, \psi_1 \right] + \phi + C(t) \tag{3.23}$$

where

$$\frac{\partial^2 \phi(x_1, x_2, x_3, t)}{\partial x_i \, \partial x_i} = \frac{\partial^2 \psi_1 \, (x_1, x_2, x_3)}{\partial x_i \, \partial x_i} = 0 \tag{3.24}$$

$$\nu_i \frac{\partial \psi_1}{\partial x_i} = \frac{2 \, x_3 \nu_2}{L^2} \qquad \text{(over all surfaces)} \tag{3.25}$$

$$\nu_i \frac{\partial \phi}{\partial x_i} = 0 \qquad \text{(over rigid boundaries)} \tag{3.26}$$

$$\left(\frac{\partial \phi}{\partial x_3}\right)_{x_3 = L} = - \ddot{\eta} \tag{3.27}$$

Again the dynamic condition used to determine η is that the pressure be independent of position on the free surface. Thus from (3.23)

$$\left(\frac{p}{\rho}\right)_{x_3 = L+\eta} = C(t) - \alpha_3 L + \left| -\alpha_2 x_2 - \alpha_3 \eta - \dot{\omega}_1 \left| (\eta + L - L_1) x_2 - L^2 \psi_1 \right| + \phi \right| \tag{3.28}$$

where, to satisfy the dynamic condition, the bracketed term vanishes. Use the same assumptions made after (2.42) to write the bracketed expression in (3.28) as

$$\alpha_3 \eta = - \alpha_2 x_2 - \dot{\omega}_1 \left| (L - L_1) x_2 - L^2 \psi_1 \right| + \phi \tag{3.29}$$

where ψ_1 and ϕ are evaluated at $x_3 = L$.

Assume

$$\eta(x_1, x_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn}(t) \; \phi_{mn}(x_1, x_2, L) \tag{3.30}$$

$$\phi(x_1, x_2, x_3, t) = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{L}{K_{mn}} \ddot{\xi}_{mn}(t) \; \phi_{mn}(x_1, x_2, x_3) \tag{3.31}$$

Introduce (3.30) and (3.31) into (3.29), and expand the remainder of (3.29) in a series of $\phi_{mn}(x_1, x_2, L)$. Then use the results of Appendix B to write

$$\ddot{\xi}_{mn} + \alpha_3 \frac{K_{mn}}{L} \xi_{mn} = -K_{mn} b_{mn} \alpha_2 + K_{mn} \dot{\omega}_1 \left| L_1 b_{mn} - L(b_{mn} - h_{mn}) \right| \tag{3.32}$$

With this formulation the pressure will be written as

$$\frac{p}{\rho} = C(t) - \alpha_2 x_2 - \alpha_3 x_3 - \dot{\omega}_1 \left| (x_3 - L_1) x_2 - L^2 \psi_1 \right| - L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\phi_{mn}}{K_{mn}} \ddot{\xi}_{mn} \tag{3.33}$$

or in terms of the coordinate system x_i'

$$\frac{p'}{\rho} = C(t) - \alpha_2 x_2' - \alpha_3 (x_3' - L_1) - \dot{\omega}_1 \left| x_2'(x_3' - 2 L_1) - L^2 \psi_1 (x_1', x_2', x_3' - L_1) \right|$$

$$- L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\phi_{mn} (x_1', x_2', x_3' - L_1)}{K_{mn}} \ddot{\xi}_{mn} \qquad (3.34)$$

The forces on the liquid tank system have the components, referred to the x_i' -axis

$$F_i' = \int_S p' \nu_i' \, dS = \int_S p' \nu_j' \frac{\partial x_i'}{\partial x_j'} \, dS$$

$$= \int_S x_i' \nu_j' \frac{\partial p'}{\partial x_j'} \, dS \qquad \text{(Theorem III)}$$

$$= \int_V \frac{\partial p'}{\partial x_i'} \, dV \qquad \text{(Theorem I)} \qquad (3.35)$$

Use the value of the pressure from (3.34) in (3.35) and integrate (the details are given in Appendix E) to obtain

$$F_i' = - \alpha_3 M \delta_{i3} - M \alpha_2 \delta_{i2} + L_1 M \dot{\omega}_1 \delta_{i2}$$

$$- M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\xi}_{mn} \gamma_{mn} \left[a_{mn} \delta_{i1} + b_{mn} \delta_{i2} \right] \qquad (3.36)$$

The moments may be calculated from

$$T_i' = \int_S \epsilon_{ijk} x_j' \nu_k' p' \, dS$$

$$= \int_V \epsilon_{ijk} x_j' \frac{\partial p'}{\partial x_k'} \, dV \qquad \text{(Theorem V)} \qquad (3.37)$$

Substitute the value of the pressure from (3.34) into (3.37) to obtain (see Appendix E for the details)

$$T_i' = M L_1 \alpha_2 \delta_{i1} - \dot{\omega}_1 \left[I_{11}' \delta_{i1} - I_{21}' \delta_{i2} - I_{31}' \delta_{i3} \right] - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} \left\{ \left[\alpha_3 b_{mn} \xi_{mn} \right.\right.$$

$$\left. + \left[L (b_{mn} - h_{mn}) - L_1 b_{mn} \right] \ddot{\xi}_{mn} \right] \delta_{i1} + \left[- \alpha_3 a_{mn} \xi_{mn} + \left[L (a_{mn} \right.\right.$$

$$\left.\left. - d_{mn}) + L_1 a_{mn} \right] \ddot{\xi}_{mn} \right] \delta_{i2} + \left[L (e_{mn} - f_{mn}) \ddot{\xi}_{mn} \right] \delta_{i3} \Big\} \qquad (3.38)$$

where

$$\left.\begin{aligned}
I_{11}' &= \rho \int_{UV} (x_2^2 + x_3^2)\, dV - 4\rho \int_{UV} x_3^2\, dV + 2\rho L^2 \int_{US} x_3\, \nu_2 \psi_1\, dS + L_1^2 M \\
I_{21}' &= \rho \int_{UV} x_1\, x_2\, dV - 2\rho L^2 \int_{US} x_1\, \nu_3\, \psi_1\, dS \\
I_{31}' &= \rho \int_{UV} x_1\, x_3\, dV - 2\rho L^2 \int_{US} x_2\, \nu_1\, \psi_1\, dS
\end{aligned}\right\} \quad (3.39)$$

If the tank has symmetry about the axis of constant acceleration, then referred to cylindrical coordinates as in Section 3.2

$$a_{mn} = d_{mn} = e_{mn} = f_{mn} = 0$$

$$I_{21}' = I_{31}' = 0$$

Thus, (3.39) reduces to

$$T_1' = \alpha_2 M L_1 - \dot{\omega}_1 I_{11}' - M \sum_{n=1}^{\infty} \gamma_n \left| \alpha_3 b_n \xi_n + \left[L(b_n - h_n) - L_1 b_n \right] \ddot{\xi}_n \right| \quad (3.40)$$

(3.36) reduces to

$$F_i' = -\alpha_3 M\, \delta_{i3} - \alpha_2 M \delta_{i2} + \dot{\omega}_1 L_1\, M \delta_{i2} - M \sum_{n=1}^{\infty} b_n \gamma_n \ddot{\xi}_n\, \delta_{i2} \quad (3.41)$$

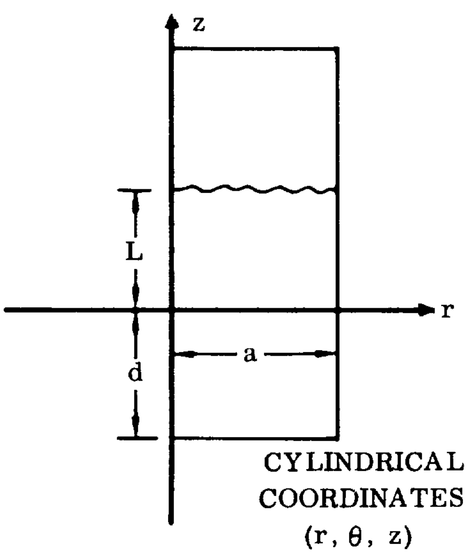

Figure 3. Cylindrical Tank

3.3.2 Parameters for a Cylindrical Tank. The analysis of the previous section is reduced for a right cylindrical tank of radius a. In Figure 3 and the following analysis,

- d = distance from the c.g. of the liquid to the tank bottom
- L = distance from the c.g. of the liquid to the free surface
- L_1 = distance from the c.g. of the liquid to an arbitrary point along the missile axis

$$\lambda_n \sim \text{roots of } J_1'(\lambda_n a) = 0, \; h = L + d \tag{3.42}$$

J_1, I_1 ~ Bessel functions of the first and second kinds, of order one

$$\dot{\omega}_1 = -\ddot{\vartheta}$$

The solution of (3.10) subject to (3.11) is

$$\psi_1 = \frac{\sin\theta}{L^2}\left\{(L-d)\,r + \frac{4}{h}\sum_{n=1}^{\infty} \frac{\left|(-1)^n - 1\right| I_1\left(\frac{n\pi r}{h}\right)\cos\left(\frac{n\pi}{h}\right)(z+d)}{\left(\frac{n\pi}{h}\right)^3 I_1'\left(\frac{n\pi}{h}a\right)}\right\} \tag{3.43}$$

From (3.12), (3.13), and (3.14)

$$\Phi_n = \cosh\lambda_n (z+d)\; J_1(\lambda_n r) \tag{3.44}$$

$$K_n = L\,\lambda_n \tanh\lambda_n h \tag{3.45}$$

Thus from (3.40) and (3.41)

$$F_3' = -\alpha_3 M \tag{3.46}$$

$$F_2' = -M\alpha_2 - L_1 M\ddot{\vartheta} - M\sum_{n=1}^{\infty} b_n \gamma_n \ddot{\xi}_n \tag{3.47}$$

$$T_1' = M L_1 \alpha_2 + I_{11}'\ddot{\vartheta} - M\sum_{n=1}^{\infty} \gamma_n \Big[\alpha_3 b_n \xi_n + \left|L(b_n - h_n) - L_1 b_n\right|\ddot{\xi}_n\Big] \tag{3.48}$$

where

$$\ddot{\xi}_n + \frac{\alpha_3 K_n}{L}\xi_n = -K_n b_n \alpha_2 + K_n\left[L(b_n - h_n) - L_1 b_n\right]\ddot{\vartheta} \tag{3.49}$$

The constants in these equations are given by

$$\gamma_n = \frac{L}{2h}\left(1 - \frac{1}{\lambda_n^2 a^2}\right)\left[\cosh\lambda_n h\, J_1(\lambda_n a)\right]^2 \tag{3.50}$$

$$\gamma_n b_n = \frac{\cosh\lambda_n h\, J_1(\lambda_n a)}{a h \lambda_n^2} \tag{3.51}$$

$$V\gamma_n K_n h_n = \frac{2\pi a}{\lambda_n} J_1 (\lambda_n a) \left[L \sinh \lambda_n h - \left(\frac{\cosh \lambda_n h - 1}{\lambda_n}\right)\right] \tag{3.52}$$

$$\frac{I_{11}}{\rho} = \pi \left[\frac{a^4 (L+d)}{4} - (L^3 + d^3) a^2\right] + 2\pi a \left[\frac{a (L-d)(L^2 - d^2)}{2}\right.$$

$$\left. + \frac{4}{h} \sum_{n=1}^{\infty} \frac{\left|(-1)^n - 1\right|^2 I_1\left(\frac{n\pi a}{h}\right)}{\left(\frac{n\pi}{h}\right)^5 I_1'\left(\frac{n\pi a}{h}\right)}\right] \tag{3.53}$$

The surface wave height is given by

$$\eta = \sin\theta \sum_{n=1}^{\infty} \xi_n(t) \cosh \lambda_n h \, J_1 (\lambda_n r) \tag{3.54}$$

3.4 MECHANICAL SYSTEMS. In general, Lagrange's equations for a moving coordinate system are, according to Kirchoff

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_i}\right) + \epsilon_{ijk} \omega_j \frac{\partial T}{\partial u_k} = F_i \tag{3.55}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_i}\right) + \epsilon_{ijk} \omega_j \frac{\partial T}{\partial \omega_k} + \epsilon_{ijk} u_j \frac{\partial T}{\partial u_k} = M_i \tag{3.56}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} = Q_n \tag{3.57}$$

where u_i is the ith component of the velocity of the origin of the moving reference frame measured along the moving axes; ω_i is the ith component of the angular velocity of the body measured along the moving axes; T is the kinetic energy of the system, i.e.

$$T = \frac{1}{2}\sum m (u_i + \epsilon_{ijk} \omega_j x_k + \dot{x}_i)(u_i + \epsilon_{i\ell m} \omega_\ell x_m + \dot{x}_i) \tag{3.58}$$

x_i is the ith component of the displacement of a mass particle measured relative to the moving reference system - it is presumed that the relative coordinates, x_i, can be expressed in terms of generalized coordinates and time, i.e.

$$x_i = x_i (q_1, q_2, q_3, \ldots . q_n, t) \tag{3.59}$$

F_i is the ith component of the force acting on the mass particle measured along the moving axes; M_i is the ith component of the moment action on the mass particle measured along the moving axes; each q_n is a generalized coordinate which describes a possible independent configurational state of the mass particle; and Q_n is the generalized force associated with q_n.

Substitute the expression for the kinetic energy into Lagrange's equations to obtain, in general

$$\sum m \Big\{ \dot{u}_i + \epsilon_{ijk}\,\omega_j\,u_k + \epsilon_{ijk}\,\dot{\omega}_j\,x_k + \omega_i\,\omega_j\,x_j - x_i\,\omega_j\,\omega_j + 2\,\epsilon_{ijk}\,\omega_j\,\dot{x}_k + \ddot{x}_i \Big\} = F_i \tag{3.60}$$

$$\sum m \Big\{ \epsilon_{ijk}\,x_j\,\dot{u}_k + \epsilon_{ijk}\,x_j\,\ddot{x}_k + 2\,\omega_i\,\dot{x}_j\,x_j - 2\,\dot{x}_i\,\omega_j\,x_j + \omega_i\,u_j\,x_j - u_i\,\omega_p\,x_p - x_p\,\omega_p\,\epsilon_{ijk}\,\omega_j\,x_k + \dot{\omega}_i\,x_j\,x_j - x_i\,\dot{\omega}_j\,x_j \Big\} = M_i \tag{3.61}$$

For motion in a plane, set $\omega_2 = \omega_3 = u_1 = x_1 = 0$. Equations (3.55) and (3.56) reduce to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_2}\right) - \omega_1\,\frac{\partial T}{\partial u_3} = F_2 \tag{3.62}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_3}\right) + \omega_1\,\frac{\partial T}{\partial u_2} = F_3 \tag{3.63}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + u_2\,\frac{\partial T}{\partial u_3} - u_3\,\frac{\partial T}{\partial u_2} = M_1 \qquad \text{(see Reference 3, p. 528)} \tag{3.64}$$

Note that (3.62), (3.63), and (3.64) are acting on the mechanical system. Thus, these quantities are the negatives of the forces and moments produced by the system. This property will be used when the equations for the mechanical systems are matched with the forces on the liquid-tank system.

3.4.1 Pendulum Analogy. To duplicate the hydrodynamic equations by a system composed of a rigid mass and a number of pendulums, it is necessary to derive the equations describing such a system in terms of an accelerating coordinate system. Consider the motion of a pendulum of mass m_n and a rigid mass of m_o as shown in Figure 4. In this system, with the x_1 axis out of the plane of the paper,

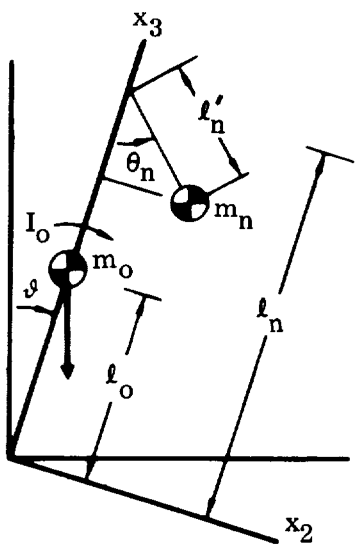

Figure 4. Pendulum

$$x_2^o = 0$$

$$x_2^n = \ell_n' \sin\theta_n$$

$$x_3^o = \ell_o$$

$$x_3^n = \ell_n - \ell_n' \cos\theta_n$$

$$\omega_1 = -\dot{\vartheta}$$

Thus, the kinetic energy is given by

$$
\begin{aligned}
T = {} & \frac{1}{2} m_o \Big\{ \big[u_2 + \ell_o \dot{\vartheta}\big]^2 + u_3^2 \Big\} + \frac{1}{2} I_o (\dot{\vartheta})^2 \\
& + \frac{1}{2} m_n \Big\{ \big[u_2 + \dot{\vartheta}(\ell_n - \ell_n' \cos\theta_n) + \ell_n' \dot{\theta}_n \cos\theta_n\big]^2 \\
& + \big[u_3 - \dot{\vartheta} \ell_n' \sin\theta_n + \ell_n' \sin\theta_n \dot{\theta}_n\big]^2 \Big\}
\end{aligned}
\tag{3.65}
$$

The equation of motion obtained by using the generalized coordinate, θ_n, is obtained from (3.57).

Thus

$$
\begin{aligned}
& \Big[2 \Big\{ u_2 + \dot{\vartheta}(\ell_n - \ell_n' \cos\theta_n) + \ell_n' \dot{\theta}_n \cos\theta_n \Big\} \Big\{ \ell_n' \sin\theta_n \dot{\theta}_n \Big\} + \Big\{ \dot{\vartheta} \ell_n' \sin\theta_n \dot{\theta}_n \\
& - \ell_n' (\dot{\theta}_n)^2 \sin\theta_n \Big\} \ell_n' \cos\theta_n + \Big\{ -\ddot{\vartheta} \ell_n' \sin\theta_n - \dot{\vartheta} \ell_n' \cos\theta_n \dot{\theta}_n + \ell_n' \ddot{\theta}_n \sin\theta_n \\
& + \ell_n' (\dot{\theta}_n)^2 \cos\theta_n \Big\} \ell_n' \sin\theta_n - \Big\{ +\dot{\vartheta}(\ell_n - \ell_n' \cos\theta_n) + \ell_n' \dot{\theta}_n \cos\theta_n \Big\} \ell_n' \dot{\vartheta} \sin\theta_n \\
& + \Big\{ -\ell_n' \dot{\vartheta} \sin\theta_n + \ell_n' \dot{\theta}_n \sin\theta_n \Big\} \Big\{ \ell_n' \dot{\vartheta} \cos\theta_n \Big\} \Big] + \dot{u}_2 \ell_n' \cos\theta_n + u_3 \ell_n' \dot{\vartheta} \cos\theta_n \\
& + \dot{u}_3 \ell_n' \sin\theta_n - u_2 \ell_n' \dot{\vartheta} \sin\theta_n + \Big\{ \ddot{\vartheta} (\ell_n - \ell_n' \cos\theta) \\
& + \ell_n' \ddot{\theta}_n \cos\theta_n \Big\} \ell_n' \cos\theta_n = \frac{Q_n}{m_n}
\end{aligned}
\tag{3.66}
$$

For small oscillations, assume θ_n and ϑ to be small, such that $\sin\theta_n \approx \theta_n$, $\cos\theta_n \approx 1$, and the products of θ_n and ϑ and their derivatives can be neglected because of their smallness in comparison with the quantities themselves. Thus, the bracketed part of (3.66) vanishes and

$$m_n \ell_n' \left[(\dot{u}_2 + \dot{\vartheta} u_3) + (\dot{u}_3 - \dot{\vartheta} u_2)\theta_n + (\ell_n - \ell_n')\ddot{\vartheta} + \ell_n' \ddot{\theta}_n \right] = Q_n \tag{3.67}$$

From (3.62), (3.63), and (3.64)

$$\begin{aligned}(m_o + m_n)(\dot{u}_2 + \dot{\vartheta} u_3) + \left[m_o \ell_o + m_n (\ell_n - \ell_n' \cos\theta) \right] \ddot{\vartheta} + m_n \ell_n' \ddot{\theta}_n \cos\theta_n - \ell_n' \sin\theta_n (\dot{\vartheta})^2 \\ + 2\ell_n' \sin\theta_n \dot{\vartheta} \dot{\theta}_n - \ell_n' \sin\theta_n (\dot{\theta}_n)^2 = F_2\end{aligned} \tag{3.68}$$

$$\begin{aligned}(m_o + m_n)(\dot{u}_3 - \dot{\vartheta} u_2) - \Big[m_o \ell_o (\dot{\vartheta})^2 + \ddot{\vartheta} \ell_n' \sin\theta_n + (\dot{\vartheta})^2 (\ell_n - \ell_n' \cos\theta) \\ + 2\dot{\vartheta} \ell_n' \cos\theta_n \dot{\theta}_n - \ell_n' \left\{ \sin\theta_n \ddot{\theta}_n + \cos\theta_n (\dot{\theta}_n)^2 \right\} \Big] = F_3\end{aligned} \tag{3.69}$$

$$\begin{aligned}- \left\{ m_o \ell_o + m_n (\ell_n - \ell_n' \cos\theta_n) \right\} \left\{ \dot{u}_2 + \dot{\vartheta} u_3 \right\} - \Big\{ m_o \ell_o^2 + m_n \Big[(\ell_n - \ell_n' \cos\theta_n)^2 \\ + \ell_n' \sin^2\theta_n \Big] + I_o \Big\} \ddot{\vartheta} + m_n \ell_n' (\dot{u}_3 - \dot{\vartheta} u_2) \sin\theta_n - m_n (\ell_n \\ - \ell_n' \cos\theta_n)(\ell_n' \cos\theta_n) \ddot{\theta}_n + m_n \Big\{ \left(\sin\theta_n \ddot{\theta}_n + \cos\theta_n (\dot{\theta}_n)^2 \right) \ell_n' \sin\theta_n \\ + (\ell_n - \ell_n' \cos\theta_n) \ell_n' \sin\theta_n (\dot{\theta}_n)^2 - 2\dot{\vartheta} \Big\{ (\ell_n')^2 \sin\theta_n \cos\theta_n \dot{\theta}_n \\ + (\ell_n - \ell_n' \cos\theta_n) \ell_n' \sin\theta_n \dot{\theta}_n \Big\} \Big\} = M_1\end{aligned} \tag{3.70}$$

Linearize (3.68), (3.69), and (3.70) to obtain

$$m_n \ell_n' \left[(\dot{u}_2 + \dot{\vartheta} u_3) + (\dot{u}_3 - \dot{\vartheta} u_2)\theta_n + (\ell_n - \ell_n')\ddot{\vartheta} + \ell_n' \ddot{\theta}_n \right] = 0 \tag{3.71}$$

$$(m_o + m_n)(\dot{u}_3 - \dot{\vartheta} u_2) = F_3 \tag{3.72}$$

$$(m_o + m_n)(\dot{u}_2 + \dot{\vartheta} u_3) + \left[m_o \ell_o + m_n (\ell_n - \ell_n') \right] \ddot{\vartheta} + m_n \ell_n' \ddot{\theta}_n = F_2 \tag{3.73}$$

$$\begin{aligned}- \left[m_o \ell_o + m_n (\ell_n - \ell_n') \right] \left[\dot{u}_2 + \dot{\vartheta} u_3 \right] - \left[m_o \ell_o^2 + m_n (\ell_n - \ell_n')^2 + I_o \right] \ddot{\vartheta} \\ - m_n (\ell_n - \ell_n') \ell_n' \ddot{\theta}_n + m_n \ell_n' (\dot{u}_3 - \dot{\vartheta} u_2) \theta_n = M_1\end{aligned} \tag{3.74}$$

For N pendulums, the equations become

$$u_2 + \dot{\vartheta} u_3 + (\dot{u}_3 - \dot{\vartheta} u_2)\theta_n + (\ell_n - \ell_n')\ddot{\vartheta} + \ell_n' \ddot{\theta}_n = 0 \quad (n = 1, 2, \dots, N) \tag{3.75}$$

$$\left(m_o + \sum_{n=1}^{N} m_n\right)(\dot{u}_3 - \dot{\vartheta} u_2) = F_3 \tag{3.76}$$

$$\left(m_o + \sum_{m=1}^{N} m_n\right)(\dot{u}_2 + \dot{\vartheta} u_3) + \left(m_o \ell_o + \sum_{n=1}^{N} m_n \left[\ell_n - \ell_n'\right]\right)\ddot{\vartheta} + \sum_{n=1}^{N} m_n \ell_n' \ddot{\theta}_n = F_2 \tag{3.77}$$

$$-\left(m_o \ell_o + \sum_{n=1}^{N} m_n \left[\ell_n - \ell_n'\right]\right)(\dot{u}_2 + \dot{\vartheta} u_3) - \left(m_o \ell_o^2 + \sum_{n=1}^{N} m_n \left[\ell_n - \ell_n'\right]^2 + I_o\right)\ddot{\vartheta}$$

$$- \sum_{n=1}^{N} m_n \left[\ell_n - \ell_n'\right] \ell_n' \ddot{\theta}_n + \sum_{n=1}^{N} m_n \ell_n' \theta_n (\dot{u}_3 - \dot{\vartheta} u_2) = M_1 \tag{3.78}$$

From (3.32), (3.40), and (3.41), the hydrodynamic behavior is described by

$$\frac{1}{K_n b_n}\ddot{\xi}_n + \alpha_3 \frac{\xi_n}{L b_n} + \alpha_2 + \frac{L_1 b_n - L(b_n - h_n)}{b_n}\ddot{\vartheta} = 0 \tag{3.79}$$

$$-M\alpha_3 = F_3' \tag{3.80}$$

$$-M\alpha_2 - M L_1 \ddot{\vartheta} - M \sum_{n=1}^{\infty} b_n \gamma_n \ddot{\xi}_n = F_2' \tag{3.81}$$

$$M L_1 \alpha_2 + I_{11}' \ddot{\vartheta} - M \sum_{n=1}^{\infty} \gamma_n \left\{\alpha_3 b_n \xi_n + \left[L(b_n - h_n) - L_1 b_n\right]\ddot{\xi}_n\right\} = T_1' \tag{3.82}$$

Thus the force, moment, and surface wave height terms will match for a finite number of pendulums if the following associations are made

$$\dot{u}_3 - \dot{\vartheta} u_2 \sim \alpha_3$$

$$\dot{u}_2 + \dot{\vartheta} u_3 \sim \alpha_2$$

$$m_o + \sum_{n=1}^{N} m_n \sim M$$

$$m_o \ell_o + \sum_{n=1}^{N} m_n (\ell_n - \ell_n') \sim M L_1$$

$$\left.\begin{aligned}
&m_o \ell_o^2 + \sum_{n=1}^{N} m_n (\ell_n - \ell_n')^2 + I_o \sim I_{11}' \\
&\theta_n \sim \frac{\xi_n}{L b_n} \\
&\ell_n' \sim \frac{L}{K_n} \\
&\ell_n \sim \frac{L_1 b_n - L(b_n - h_n)}{b_n} + \frac{L}{K_n} \\
&m_n \sim M \gamma_n b_n^2 K_n \\
&m_o \sim M\Big(1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n\Big) \\
&\ell_o \sim \frac{L_1 - \sum_{n=1}^{N} \gamma_n b_n K_n \left|L_1 b_n - L(b_n - h_n)\right|}{1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n} \\
&I_o \sim I_{11}' - m_o \ell_o^2 - \sum_{n=1}^{N} m_n (\ell_n - \ell_n')^2
\end{aligned}\right\} \qquad (3.83)$$

3.4.2 Spring-Mass Analogy. Consider also a spring-mass mechanical system as shown in Figure 5. The kinetic energy, T, of this system is given by

$$\begin{aligned}
T = &\frac{1}{2} m_o \left| u_3^2 + (u_2 + \dot{\vartheta} \ell_o)^2 \right| \\
&+ \frac{1}{2} m_n \left| (u_3 - \dot{\vartheta} x_n)^2 \right. \\
&\left. + (u_2 + \dot{x}_n + \dot{\vartheta} \ell_n)^2 \right| \\
&+ \frac{1}{2} I_o \dot{\vartheta}^2
\end{aligned} \qquad (3.84)$$

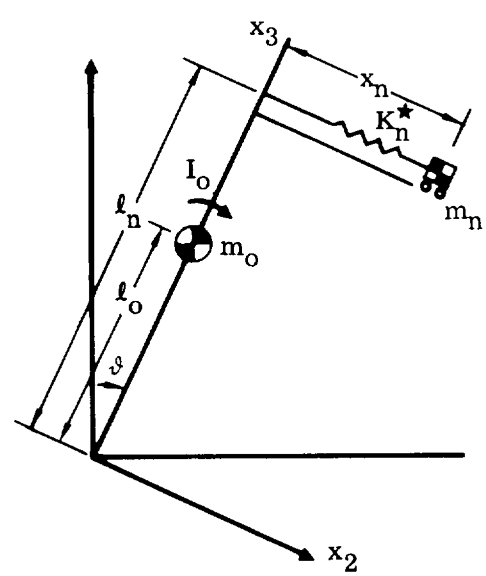

Figure 5. Spring-Mass

From Kirchoff's relations, the linearized equations are

$$(m_o + m_n)(\dot{u}_2 + \dot{\vartheta} u_3) + (m_o \ell_o + m_n \ell_n)\ddot{\vartheta} + m_n \ddot{x}_n = F_2 \qquad \text{(for each n)} \tag{3.85}$$

$$(m_o + m_n)(\dot{u}_3 - \dot{\vartheta} u_2) = F_3 \tag{3.86}$$

$$-(m_o \ell_o + m_n \ell_n)(\dot{u}_2 + \dot{\vartheta} u_3) - (m_o \ell_o^2 + m_n \ell_n^2 + I_o)\ddot{\vartheta}$$

$$- m_n \ell_n \ddot{x}_n - m_n x_n (\dot{u}_3 - \dot{\vartheta} u_2) = M_1 \tag{3.87}$$

To obtain the last equation, let x_n be the generalized coordinate, and include the potential energy term $\frac{K_n^{\star} x_n^2}{2}$. Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_n}\right) - \frac{\partial T}{\partial x_n} + K_n^{\star} x_n = 0$$

$$m_n \left| (\dot{u}_2 + \dot{\vartheta} u_3) + \ddot{x}_n + \ell_n \ddot{\vartheta} \right| + K_n^{\star} x_n = 0 \tag{3.88}$$

Take a finite number of spring-mass elements; the equations become

$$(\dot{u}_2 + \dot{\vartheta} u_3) + \ddot{x}_n + \ell_n \ddot{\vartheta} + \frac{K_n^{\star} x_n}{m_n} = 0 \qquad (n = 1, 2, \ldots N) \tag{3.89}$$

$$(m_o + \sum_{n=1}^{N} m_n)(\dot{u}_3 - \dot{\vartheta} u_2) = F_3 \tag{3.90}$$

$$(m_o + \sum_{n=1}^{N} m_n)(\dot{u}_2 + \dot{\vartheta} u_3) + (m_o \ell_o + \sum_{n=1}^{N} m_n \ell_n)\ddot{\vartheta} + \sum_{n=1}^{N} m_n \ddot{x}_n = F_2 \tag{3.91}$$

$$-(m_o \ell_o + \sum_{n=1}^{N} m_n \ell_n)(\dot{u}_2 + \dot{\vartheta} u_3) - (m_o \ell_o^2 + \sum_{n=1}^{N} m_n \ell_n^2 + I_o)\ddot{\vartheta}$$

$$- \sum_{n=1}^{N} m_n \ell_n \ddot{x}_n + \sum_{n=1}^{N} m_n x_n (\dot{u}_3 - \dot{\vartheta} u_2) = M_1 \tag{3.92}$$

Thus, the forces, moments, and surface wave height will match the mechanical system's equations if

$$\mathring{u}_3 - \mathring{\vartheta} u_2 \sim \alpha_3$$

$$\dot{u}_2 + \dot{\vartheta} u_3 \sim \alpha_2$$

$$m_o + \sum_{n=1}^{N} m_n \sim M$$

$$m_o \ell_o + \sum_{n=1}^{N} m_n \ell_n \sim M L_1$$

$$m_o \ell_o^2 + \sum_{n=1}^{N} m_n \ell_n^2 + I_o \sim I_{11}'$$

$$x_n \sim \frac{\xi_n}{K_n b_n}$$

$$\ell_n \sim \frac{L_1 b_n - L(b_n - h_n)}{b_n}$$

$$m_n \sim M \gamma_n b_n^2 K_n$$

$$K_n^{\star} \sim \frac{\alpha_3}{L} M \gamma_n b_n^2 K_n^2$$

$$m_o \sim M\left(1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n\right)$$

$$\ell_o \sim \frac{L_1 - \sum_{n=1}^{N} \gamma_n b_n K_n \left[L_1 b_n - L(b_n - h_n)\right]}{1 - \sum_{n=1}^{N} \gamma_n b_n^2 K_n}$$

$$I_o \sim I_{11}' - m_o \ell_o^2 - \sum_{n=1}^{N} m_n \ell_n^2$$

SECTION 4

CONCLUSION

In this report the hydrodynamic equations describing the dynamic behavior of liquids contained in tanks of arbitrary shape are derived for a missile with six degrees of freedom -- three rotational and three translational. These results are summarized in Section 2.2.3.

For tne special case of three degrees of freedom (one translational component along an axis perpendicular to the axis of constant acceleration and one rotational component about an axis perpendicular to the two axes previously mentioned) the equations are simplified and given in Section 3.

In order to incorporate the dynamic liquid behavior into an analysis of the entire missile for control purposes, the pressure, forces, and moments are rewritten using the fact that the coordinate system is fixed in the missile but is no longer located at the center of gravity of the liquid. This is done in Section 3.3.1 for a tank that possesses symmetry about the axis of constant acceleration but that is otherwise of arbitrary shape. In Section 3.3.2 the appropriate quantities are given for a circular cylindrical tank.

The equations of motion of two mechanical systems are presented in Section 3.4. The parameters from the hydrodynamic solution are matched with the parameters of the mechanical systems so that the corresponding equations will be identical.

This report is the basis of companion report GD|A-DDE64-062, which describes the digital computer routine used to obtain the hydrodynamic parameters for tanks of arbitrary shape.

SECTION 5

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APPENDIX A

ROTATING COORDINATE SYSTEMS

Assume that a rigid body is rotating about an axis in space. Let this axis be designated by a unit vector which has components λ_i when referred to the Cartesian coordinate system $y_i^\star$, fixed in space. Let $P{:}y_i^\star$ be the coordinates of a point at time $t = t_1$ (see Figure A). Let the body rotate about axis λ_i through an angle $\delta\theta$ in time δt. The point P will move to $P'{:}y_i'$, which lies on the arc of a circle with the center at $C{:}\xi_i$ and having a radius of $|CP| = |CP'|$. The components of a unit vector along the directed line segment CP are

$$\mu_i = \frac{y_i^\star - \xi_i}{|CP|} \qquad (A.1)$$

Also, the components of a unit vector along the directed line segment CP′ are

$$\eta_i = \frac{y_i' - \xi_i}{|CP|} \qquad (A.2)$$

Draw a line from P′ perpendicular to the line CP, intersecting CP at N. The components of a vector along the directed line segment NP′ are, since it will be perpendicular to both the axis of rotation and CP

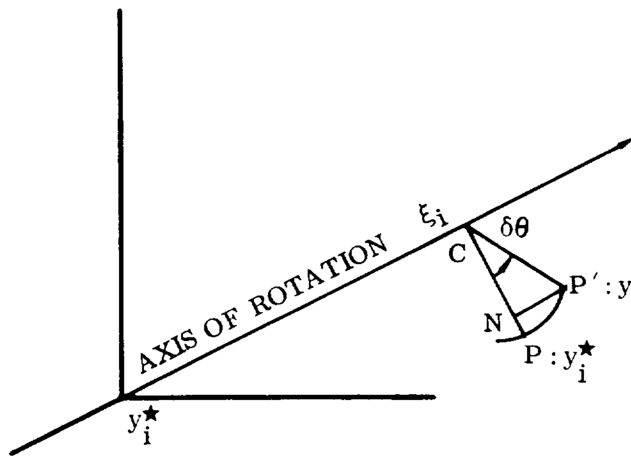

Figure A. Rotating Coordinates

$$\nu_i = \epsilon_{ijk} \lambda_j \mu_k \qquad (A.3)$$

It is also true that the direction cosines obey the laws

$$\lambda_i \mu_i = 0 = \lambda_i (y_i^\star - \xi_i) \qquad (A.4)$$

Since ξ_i are the coordinates of a point on the line with direction cosines λ_i, it follows that

$$\frac{\xi_1}{\lambda_1} = \frac{\xi_2}{\lambda_2} = \frac{\xi_3}{\lambda_3}$$

i.e.

$$\xi_k = \lambda_k \frac{\xi_m}{\lambda_m} \qquad \text{(no sum on m, m = 1, 2 or 3)} \qquad \text{(A.5)}$$

From (A.4)

$$\lambda_i \, \xi_i = \lambda_i \, y_i^{\star} \qquad \text{(A.6)}$$

Substitute from (A.5) into (A.6) to obtain

$$\lambda_i \, \lambda_i \frac{\xi_m}{\lambda_m} = \lambda_i \, y_i^{\star} \qquad \text{(no sum on m)} \qquad \text{(A.7)}$$

i.e., since $\lambda_i \, \lambda_i = 1$

$$\xi_m = \lambda_m \, \lambda_i \, y_i^{\star} \qquad \text{(A.8)}$$

The coordinates of the point P′, $|CP| \, \eta_i$, may be written as (see Figure A)

$$|CP| \, \eta_i = |CP| \cos \delta\theta \, \mu_i + |CP| \sin \delta\theta \, \nu_i \qquad \text{(A.9)}$$

Thus, using (A.1), (A.2), and (A.3)

$$y_i' - \xi_i = (y_i^{\star} - \xi_i) \cos \delta\theta + \epsilon_{ijk} \, \lambda_j \, (y_k^{\star} - \xi_k) \sin \delta\theta \qquad \text{(A.10)}$$

i.e.

$$\delta y_i^{\star} \equiv y_i' - y_i^{\star} = y_i^{\star}(\cos \delta\theta - 1) + \xi_i \, (1 - \cos \delta\theta) - \epsilon_{ijk} \, \lambda_j \, \xi_k \sin \delta\theta$$
$$+ \epsilon_{ijk} \, \lambda_j \, y_k^{\star} \sin \delta\theta \qquad \text{(A.11)}$$

The third term on the right hand side of (A.11) may be written as

$$\epsilon_{ijk} \, \lambda_j \, \lambda_k \lambda_m \, y_m^{\star} \sin \delta\theta = 0$$

since

$$\epsilon_{ijk} \, \lambda_j \, \lambda_k = 0$$

Thus, expand $\cos \delta\theta$ and $\sin \delta\theta$ in a Maclaurin series to write (A.11) as

$$\delta y_i^{\star} = (\xi_i - y_i^{\star}) \left[\frac{(\delta\theta)^2}{2!} + \ldots\right] + \epsilon_{ijk} \, \lambda_j \, y_k^{\star} \left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \ldots\right] \qquad \text{(A.12)}$$

Divide (A.12) by δt and take the limit as δt and $\delta\theta$ approach zero to obtain

$$\frac{dy_i^\star}{dt} = \lim_{\delta t\to 0} \frac{\delta y_i^\star}{\delta t} = \epsilon_{ijk}\,\lambda_j\, y_k^\star \frac{d\theta}{dt} \tag{A.13}$$

where $\frac{d\theta}{dt} = \lim_{\delta t\to 0} \frac{\delta\theta}{\delta t}$ is the angular velocity. The components of the angular velocity referred to the system $y_i^\star$ are

$$\Omega_j = \lambda_j \frac{d\theta}{dt} \tag{A.14}$$

Thus

$$\frac{dy_i^\star}{dt} = \epsilon_{ijk}\,\Omega_j\, y_k^\star \tag{A.15}$$

Equation A.15 gives the instantaneous rate of change of a point on a rigid body due to the rotation of this body about an arbitrary axis.

Consider a rectangular Cartesian coordinate system obtained from $y_i^\star$ by a rotation

$$x_i = a_{ij}\, y_j^\star \tag{A.16}$$

The inverse transformation is

$$y_i^\star = a_{ji}\, x_j \tag{A.17}$$

where

$$a_{ij} = \cos\,(x_i,\, y_j^\star)$$

For a general rotation, $a_{ij} = a_{ij}(t)$. Differentiate the identity $a_{ik}\, a_{jk} = \delta_{ij}$ with respect to t to obtain

$$a_{ik} \frac{d}{dt} a_{jk} = -\, a_{jk} \frac{d}{dt} a_{ik} \tag{A.18}$$

The tensor in (A.18) is skew-symmetric; the following paragraph will impute a physical meaning to this tensor.

Differentiate (A.17) with respect to time and use (A.15)

$$\frac{d}{dt} y_i^\star = \left[\frac{d}{dt} a_{ji}\right] x_j + a_{ji} \frac{d}{dt} x_j = \epsilon_{ijk}\,\Omega_j\, y_k^\star \tag{A.19}$$

Let ω_i be the components of Ω_j in the coordinate system x_i; i.e.

$$\omega_j = a_{jk}\,\Omega_k \qquad \text{or} \qquad \Omega_j = a_{kj}\,\omega_k \tag{A.20}$$

Assume that the point x_j moves with the body so that

$$\frac{d}{dt}\,x_j = 0 \tag{A.21}$$

Then, from (A.17), (A.19), (A.20), and (A.21)

$$\left[\frac{d}{dt}\,a_{ji}\right] x_j - \epsilon_{ijk}\,a_{nj}\,\omega_n\,a_{pk}\,x_p = 0$$

or, changing the dummy indices

$$\left[\frac{d}{dt}\,a_{ji}\right] x_j - \epsilon_{imk}\,a_{nm}\,a_{jk}\,\omega_n\,x_j = 0 \tag{A.22}$$

Since (A.22) must be true for arbitrary x_j, it follows that

$$\frac{d}{dt}\,a_{ji} = \epsilon_{imk}\,a_{nm}\,a_{jk}\,\omega_n \tag{A.23}$$

From (A.23) it follows that

$$a_{pi}\,\frac{d}{dt}\,a_{ji} = \epsilon_{imk}\,a_{pi}\,a_{nm}\,a_{jk}\,\omega_n \tag{A.24}$$

$$= \epsilon_{pnj}\,\omega_n = -\epsilon_{pjn}\,\omega_n \tag{A.25}$$

(Note that in going from (A.24) to (A.25) use has been made of the fact that

$$\epsilon_{imk}\,a_{pi}\,a_{nm}\,a_{jk} = \begin{vmatrix} a_{p1} & a_{p2} & a_{p3} \\ a_{n1} & a_{n2} & a_{n3} \\ a_{j1} & a_{j2} & a_{j3} \end{vmatrix} = \epsilon_{pnj}\,) \tag{A.26}$$

Thus, the skew-symmetric tensor $a_{ik}\frac{d}{dt}a_{jk}$ can be written as

$$a_{ik}\,\frac{d}{dt}\,a_{jk} = -\,\epsilon_{ijk}\,\omega_k \tag{A.27}$$

APPENDIX B

ORTHOGONALITY

Assuming that the order of summation and integration may be interchanged, (2.54) becomes (after multiplying by ϕ_{ij} and integrating over the undisturbed free surface)

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{L}{K_{mn}} \ddot{\xi}_{mn}(t) + \alpha_3 \xi_{mn}(t)\right] \int_{UFS} \phi_{mn}(x_1, x_2, L)\, \phi_{ij}(x_1, x_2, L)\, dS =$$

$$- \alpha_1 \int_{UFS} x_1\, \phi_{ij}(x_1, x_2, L)\, dS - \alpha_2 \int_{UFS} x_2\, \phi_{ij}(x_1, x_2, L)\, dS$$

$$- \dot{\omega}_1 \int_{UFS} \left[x_2 L - L^2 \psi_1 (x_1, x_2, L)\right] \phi_{ij}(x_1, x_2, L)\, dS$$

$$- \dot{\omega}_2 \int_{UFS} \left[x_1 L - L^2 \psi_2 (x_1, x_2, L)\right] \phi_{ij}(x_1, x_2, L)\, dS$$

$$- \dot{\omega}_3 \int_{UFS} \left[x_1 x_2 - L^2 \psi_3 (x_1, x_2, L)\right] dS \qquad \text{(B.1)}$$

Consider

$$I = \int_{UFS} \phi_{mn}(x_1, x_2, L)\, \phi_{ij}(x_1, x_2, L)\, dS \equiv$$

$$\frac{L}{K_{mn} - K_{ij}} \int_{UFS} \left[\frac{K_{mn}}{L} \phi_{mn} \phi_{ij} - \phi_{mn} \frac{K_{ij}}{L} \phi_{ij}\right] dS \qquad \text{(B.2)}$$

(no sum on i, j, m, or n) but from (2.47), (B.2) becomes

$$I = \frac{L}{K_{mn} - K_{ij}} \int_{UFS} \left[\phi_{ij} \frac{\partial \phi_{mn}}{\partial x_3} - \phi_{mn} \frac{\partial \phi_{ij}}{\partial x_3}\right] dS$$

$$= \frac{L}{K_{mn} - K_{ij}} \int_{US} \left[\phi_{ij} \nu_k \frac{\partial \phi_{mn}}{\partial x_k} - \phi_{mn} \nu_k \frac{\partial \phi_{ij}}{\partial x_k}\right] dS \qquad \text{(B.3)}$$

The integration in (B.3) is over US, the UFS plus the rigid walls, and is valid because of (2.47). Use the divergence theorem to write (B.3) as

$$I = \frac{L}{K_{mn} - K_{ij}} \int_{UV} \frac{\partial}{\partial x_k} \left[\phi_{ij} \frac{\partial \phi_{mn}}{\partial x_k} - \phi_{mn} \frac{\partial \phi_{ij}}{\partial x_k} \right] dV$$

$$= 0, \text{ for } K_{mn} \neq K_{ij}, \text{ since } \phi_{mn} \text{ is harmonic}$$

Let

$$\left.\begin{aligned}
a_{mn} &= \frac{\int_{UFS} x_1 \phi_{mn}\, dS}{L \int_{UFS} \phi_{mn}^2\, dS} \\
b_{mn} &= \frac{\int_{UFS} x_2 \phi_{mn}\, dS}{L \int_{UFS} \phi_{mn}^2\, dS} \\
h_{mn} &= \frac{\int_{UFS} \psi_1 \phi_{mn}\, dS}{\int_{UFS} \phi_{mn}^2\, dS} \\
d_{mn} &= \frac{\int_{UFS} \psi_2 \phi_{mn}\, dS}{\int_{UFS} \phi_{mn}^2\, dS} \\
e_{mn} &= \frac{\int_{UFS} x_1 x_2 \phi_{mn}\, dS}{L^2 \int_{UFS} \phi_{mn}^2\, dS} \\
f_{mn} &= \frac{\int_{UFS} \psi_3 \phi_{mn}\, dS}{\int_{UFS} \phi_{mn}^2\, dS}
\end{aligned}\right\} \qquad \text{(B.4)}$$

Thus, (B.1) reduces to

$$\frac{L}{K_{mn}} \dot{\xi}_{mn}(t) + \alpha_3 \xi_{mn} = - \alpha_1 L a_{mn} - \alpha_2 L b_{mn}$$

$$- \dot{\omega}_1 \left[L^2 b_{mn} - L^2 h_{mn}\right] - \dot{\omega}_2 \left[L^2 a_{mn} - L^2 d_{mn}\right]$$

$$- \dot{\omega}_3 \left[L^2 e_{mn} - L^2 f_{mn}\right]$$

or

$$\dot{\xi}_{mn}(t) + \alpha_3 \frac{K_{mn}}{L} \xi_{mn} = - \alpha_1 K_{mn} a_{mn} - \alpha_2 K_{mn} b_{mn}$$

$$- L K_{mn} \left[b_{mn} - h_{mn}\right] \dot{\omega}_1 - L K_{mn} \left[a_{mn} - d_{mn}\right] \dot{\omega}_2$$

$$- L K_{mn} \left[e_{mn} - f_{mn}\right] \dot{\omega}_3 \qquad \text{(B. 5)}$$

Consider the integral

$$\int_{UFS} \psi_i \phi_{mn} \, dS = \int_{UFS} \psi_i \frac{L}{K_{mn}} \frac{\partial \phi_{mn}}{\partial x_3} \, dS \qquad \text{(no sum on m or n)}$$

$$= \frac{L}{K_{mn}} \int_{UFS} \psi_i \nu_k \frac{\partial \phi_{mn}}{\partial x_k} dS = \frac{L}{K_{mn}} \int_{US} \psi_i \nu_k \frac{\partial \phi_{mn}}{\partial x_k} dS$$

By the symmetric form of Green's theorem

$$\int_{UFS} \psi_i \phi_{mn} \, dS = \frac{L}{K_{mn}} \int_{US} \phi_{mn} \nu_k \frac{\partial \psi_i}{\partial x_k} dS \qquad \text{(B. 6)}$$

Thus, from (2.33), (2.34), and (2.35)

$$\int_{UFS} \psi_1 \phi_{mn} \, dS = \frac{2}{L K_{mn}} \int_{US} x_3 \nu_2 \phi_{mn} \, dS \qquad \text{(B. 7)}$$

$$\int_{UFS} \psi_2 \, \phi_{mn} \, dS = \frac{2}{L \, K_{mn}} \int_{US} x_1 \, \nu_3 \, \phi_{mn} \, dS \tag{B.8}$$

$$\int_{UFS} \psi_3 \, \phi_{mn} \, dS = \frac{2}{L \, K_{mn}} \int_{US} x_2 \, \nu_1 \, \phi_{mn} \, dS \tag{B.9}$$

Define

$$\gamma_{mn} = \frac{L}{V} \int_{UFS} (\phi_{mn})^2 \, dS \tag{B.10}$$

From (B.4) and (B.7) through B.9)

$$\left.\begin{aligned}
&\int_{UFS} x_1 \, \phi_{mn} \, dS = V \, a_{mn} \, \gamma_{mn} \\
&\int_{UFS} x_2 \, \phi_{mn} \, dS = V \, b_{mn} \, \gamma_{mn} \\
&\int_{US} x_3 \, \phi_{mn} \, \nu_2 \, dS = \frac{V \, K_{mn} \, h_{mn} \, \gamma_{mn}}{2} \\
&\int_{US} x_1 \, \phi_{mn} \, \nu_3 \, dS = \frac{V \, K_{mn} \, d_{mn} \, \gamma_{mn}}{2} \\
&\int_{UFS} x_1 \, x_2 \, \phi_{mn} \, dS = V \, L \, e_{mn} \, \gamma_{mn} \\
&\int_{US} x_2 \, \phi_{mn} \, \nu_1 \, dS = \frac{V \, K_{mn} \, f_{mn} \gamma_{mn}}{2}
\end{aligned}\right\} \tag{B.11}$$

APPENDIX C

FORCES

The substitution of (2.53) into (2.57) yields

$$F_i^{\star} = \rho \int_V \left[- \alpha_3 \delta_{i3} - \alpha_1 \delta_{i1} - \alpha_2 \delta_{i2} - \left(x_2 \delta_{i3} + x_3 \delta_{i2} - L^2 \frac{\partial \psi_1}{\partial x_i} \right) \dot{\omega}_1 \right.$$

$$- \left(x_1 \delta_{i3} + x_3 \delta_{i1} - L^2 \frac{\partial \psi_2}{\partial x_i} \right) \dot{\omega}_2 - \left(x_1 \delta_{i2} + x_2 \delta_{i1} - L^2 \frac{\partial \psi_3}{\partial x_i} \right) \dot{\omega}_3$$

$$\left. - L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{K_{mn}} \frac{\partial \phi_{mn}}{\partial t} \dot{\xi}_{mn} \right] dV \qquad \text{(C.1)}$$

Consider the integral of each term separately. The first one becomes

$$\rho \int_V \left[- \alpha_3 \delta_{i3} \right] dV = \rho V \left[- \alpha_3 \delta_{i3} \right]$$

since α_3 is independent of the space coordinates.

In the remaining integrals, write the integral over the actual volume as the sum of two integrals, one over the undisturbed volume and one over the difference between the actual volume and the undisturbed volume. Since each term contains a multiplicative factor of α_i or $\dot{\omega}_i$, the second integral will be neglected in keeping with the neglecting of second-order infinitesimals. Thus, the volume integrals become integrals over the undisturbed volume, and the second and third terms in (C.1) may be written as

$$\rho \int_V \left[- \alpha_1 \delta_{i1} - \alpha_2 \delta_{i2} \right] dV = - \rho V \alpha_1 \delta_{i1} - \rho V \alpha_2 \delta_{i2} \qquad \text{(C.2)}$$

Since the origin of the x_i system was chosen at the center of gravity of the fluid

$$\int_{UV} x_i \, dV = 0 \qquad \text{(C.3)}$$

Consider

$$\int_{UV} \frac{\partial \psi_k}{\partial x_i} \, dV = \int_{US} \psi_k \, \nu_i \, dS \qquad \text{(by Theorem I)}$$

$$= \int_{US} \psi_k \frac{\partial x_i}{\partial x_j} \nu_j \, dS$$

$$= \int_{US} x_i \frac{\partial \psi_k}{\partial x_j} \nu_j \, dS \qquad \text{(by Theorem III)} \qquad \text{(C.4)}$$

Therefore

$$\int_{UV} \frac{\partial \psi_1}{\partial x_i} \, dV = \int_{US} x_i \frac{2 x_3 \nu_2}{L^2} \, dS \qquad \text{(from (C.4) and (2.33))}$$

$$= \frac{2}{L^2} \int_{UV} \frac{\partial}{\partial x_2} (x_i x_3) \, dV \qquad \text{(by Theorem I)}$$

$$= \frac{2}{L^2} \int_{UV} \delta_{i2} \, x_3 \, dV = 0 \qquad \text{(C.5)}$$

Also

$$\int_{UV} \frac{\partial \psi_2}{\partial x_i} \, dV = \int_{US} x_i \frac{2 x_1 \nu_3}{L^2} \, dS \qquad \text{(from (C.4) and (2.34))}$$

$$= \frac{2}{L^2} \int_{UV} \frac{\partial}{\partial x_3} (x_i x_1) \, dV \qquad \text{(by Theorem I)}$$

$$= \frac{2}{L^2} \int_{UV} \delta_{i3} \, x_1 \, dV = 0 \qquad \text{(C.6)}$$

$$\int_{UV} \frac{\partial \psi_3}{\partial x_i} \, dV = \int_{US} x_i \frac{2 x_2 \nu_1}{L^2} \, dS \qquad \text{(from (C.4) and (2.35))}$$

$$\int_{UV} \frac{\partial\psi_3}{\partial x_i} dV = \frac{2}{L^2} \int_{UV} \frac{\partial}{\partial x_1} (x_i\, x_2)\, dV \qquad \text{(by Theorem I)}$$

$$= \frac{2}{L^2} \int_{UV} \delta_{i1}\, x_2\, dV = 0 \qquad (C.7)$$

Thus, from (C.4) through (C.7), the integrals of the fourth, fifth, and sixth terms of (C.1) vanish.

Consider

$$\int_{UV} \frac{\partial\phi_{mn}}{\partial x_i} dV = \int_{US} \phi_{mn}\, \nu_i\, dS \qquad \text{(by Theorem I)}$$

$$= \int_{US} \phi_{mn}\, \nu_j \frac{\partial x_i}{\partial x_j} dS = \int_{US} x_i\, \nu_j \frac{\partial \phi_{mn}}{\partial x_j} dS \quad \text{(by Theorem III)}$$

$$= \frac{K_{mn}}{L} \int_{UFS} x_i\, \phi_{mn}\, dS \qquad \text{(from (2.47))} \qquad (C.8)$$

Thus, from (C.8) and (B.11)

$$\int_{UV} \frac{\partial\phi_{mn}}{\partial x_1} dV = \frac{K_{mn}}{L} V a_{mn} \gamma_{mn} \qquad (C.9)$$

$$\int_{UV} \frac{\partial\phi_{mn}}{\partial x_2} dV = \frac{K_{mn}}{L} V b_{mn} \gamma_{mn} \qquad (C.10)$$

$$\int_{UV} \frac{\partial\phi_{mn}}{\partial x_3} dV = \int_{US} x_3\, \nu_j \frac{\partial\phi_{mn}}{\partial x_j} dS \qquad \text{(from (C.8))}$$

$$= \int_{US} L\, \nu_j \frac{\partial\phi_{mn}}{\partial x_j} dS \qquad \text{(from (2.47))}$$

$$= \int_{UV} L \frac{\partial^2 \phi_{mn}}{\partial x_j\, \partial x_j} dV = 0 \qquad \text{(by Theorem I and (2.47))} \qquad (C.11)$$

Substitute the values of these integrals into (C.1) to get, letting $M = \rho V$

$$F_i^{\star} = -M\,\alpha_i - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \dot{\xi}_{mn}\,\gamma_{mn}\,(a_{mn}\,\delta_{i1} + b_{mn}\,\delta_{i2}) \tag{C.12}$$

APPENDIX D

MOMENTS

Substitution of (2.53) into (2.59) yields

$$T_i = \rho \int_V \epsilon_{ijk} x_j \left[-\alpha_3 \delta_{k3} - \alpha_1 \delta_{k1} - \alpha_2 \delta_{k2} - \left(x_2 \delta_{k3} + x_3 \delta_{k2} - L^2 \frac{\partial \psi_1}{\partial x_k} \right) \dot{\omega}_1 - \left(x_1 \delta_{k3} + x_3 \delta_{k1} - L^2 \frac{\partial \psi_2}{\partial x_k} \right) \dot{\omega}_2 - \left(x_1 \delta_{k2} + x_2 \delta_{k1} - L^2 \frac{\partial \psi_3}{\partial x_k} \right) \dot{\omega}_3 - L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{K_{mn}} \frac{\partial \phi_{mn}}{\partial x_k} \ddot{\xi}_{mn} \right] dV \tag{D.1}$$

Consider the first term

$$\rho \int_V - \alpha_3 \epsilon_{ijk} x_j \delta_{k3} \, dV = - \rho \alpha_3 \int_V \epsilon_{ijk} \frac{\partial (x_j x_3)}{\partial x_k} dV$$

$$= - \rho \alpha_3 \int_S \epsilon_{ijk} x_j x_3 \nu_k \, dS \qquad \text{(by Theorem I)} \tag{D.2}$$

Write the surface integral in (D.2) as the sum of two integrals, one over the undisturbed surface and the other over the difference between the actual surface and the undisturbed surface.

$$\int_{US} \epsilon_{ijk} x_j x_3 \nu_k \, dS = \int_{UV} \epsilon_{ijk} x_j \delta_{k3} \, dV = 0 \qquad \text{(see (C.3))}$$

$$\int_{S-US} \epsilon_{ijk} x_j x_3 \nu_k \, dS = \int_{FS} \epsilon_{ijk} x_j x_3 \nu_k \, dS - \int_{UFS} \epsilon_{ijk} x_j L \nu_k \, dS$$

$$= \int_{UFS} \epsilon_{ijk} x_j \eta \nu_k \, dS \tag{D.3}$$

(In deriving (D.3) the difference between the actual liquid surface and the undisturbed liquid surface on the walls of the tank has been neglected, and, since η is considered to be small, the integral on the right side of (D.3) is over the undisturbed free surface instead of the actual free surface. This is in keeping with the previous assumption to neglect products of infinitesimals.)

Substitute the value of η from (2.49) into (D.3), and interchange the orders of integration and summation to obtain

$$-\rho\,\alpha_3 \int_{UFS} \epsilon_{ijk}\, x_j\, \eta\, \nu_k\, dS = -\rho\,\alpha_3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn} \int_{UFS} \epsilon_{ij3}\, x_j\, \phi_{mn}(x_1, x_2, L)\, dS$$

$$= -\rho\,\alpha_3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn} \left[\delta_{i1} \int_{UFS} x_2\, \phi_{mn}(x_1, x_2, L)\, dS \right.$$

$$\left. -\delta_{i2} \int_{UFS} x_1\, \phi_{mn}(x_1, x_2, L) dS\right]$$

$$= -\rho\,\alpha_3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn} \left[\delta_{i1}\, b_{mn}\, V\gamma_{mn} \right.$$

$$\left. -\delta_{i2}\, a_{mn}\, V\,\gamma_{mn}\right] \tag{D.4}$$

The integrals of the second and third terms are zero, since the origin of the x_i-system is located at the center of gravity of the liquid. The volume integrals of the remaining terms are taken over the undisturbed volume for the same reasons as given in Appendix C.

Consider the integral of the fourth term

$$I_{i,\omega_1} = \int_{UV} \left|\epsilon_{ij3}\, x_j\, x_2 + \epsilon_{ij2}\, x_j\, x_3\right| dV - L^2 \int_{UV} \epsilon_{ijk}\, x_j \frac{\partial \psi_1}{\partial x_k}\, dV$$

$$= \int_{UV} \left|\epsilon_{ij3}\, x_j\, x_2 + \epsilon_{ij2}\, x_j\, x_3\right| dV - L^2 \int_{US} \epsilon_{ijk}\, x_j\, \psi_1\, \nu_k\, dS \quad \text{(by Theorem V)}$$

Thus

$$I_{1,\omega_1} = \int_{UV} (x_2^2 - x_3^2)\, dV - L^2 \int_{US} (x_2\,\nu_3 - x_3\,\nu_2)\psi_1\, dS$$

$$= \int_{UV} (x_2^2 - x_3^2)\, dV - L^2 \int_{US} \nu_i\, x_2\, x_3 \frac{\partial \psi_1}{\partial x_i}\, dS$$

$$+ 2L^2 \int_{US} x_3\, \nu_2\, \psi_1\, dS \quad \text{(by Theorem IV)}$$

$$= \int_{UV} (x_2^2 - x_3^2)\, dV - L^2 \int_{US} x_2\, x_3 \frac{2x_3\nu_2}{L^2}\, dS$$

$$+ 2L^2 \int_{US} x_3\nu_2\, \psi_1\, dS \quad \text{(from (2.33))}$$

$$I_{1,\omega_1} = \int_{UV} (x_2^2 - x_3^2)\, dV - 2 \int_{UV} \frac{\partial}{\partial x_2} (x_2 x_3^2)\, dV$$

$$+ 2 L^2 \int_{US} x_3 \nu_2 \psi_1 \, dS \qquad \text{(by Theorem I)}$$

$$= \int_{UV} (x_2^2 + x_3^2)\, dV - 4 \int_{UV} x_3^2 \, dV + 2 L^2 \int_{US} x_3 \nu_2 \psi_1 \, dS \qquad \text{(D. 5)}$$

Define $I_{11} = \rho I_{1,\omega_1}$. Similarly

$$I_{2,\omega_1} = \int_{UV} - x_1 x_2 \, dV - L^2 \int_{US} (x_3 \nu_1 - x_1 \nu_3)\, \psi_1 \, dS$$

$$= \int_{UV} -x_1 x_2 \, dV - L^2 \int_{US} \nu_i x_3 x_1 \frac{\partial \psi_1}{\partial x_i} \, dS$$

$$+ 2 L^2 \int_{US} x_1 \nu_3 \psi_1 \, dS \qquad \text{(Theorem IV)}$$

$$= \int -x_1 x_2 \, dV - L^2 \int_{US} x_3 x_1 \frac{2 x_3 \nu_2}{L^2} \, dS$$

$$+ 2 L^2 \int_{US} x_1 \nu_3 \psi_1 \, dS \qquad \text{(from (2.33))}$$

$$= - \int_{UV} x_1 x_2 \, dV + 2 L^2 \int_{US} x_1 \nu_3 \psi_1 \, dS \qquad \text{(Theorem I)} \qquad \text{(D. 6)}$$

Define $I_{21} = -\rho I_{2,\omega_1}$. Finally

$$I_{3,\omega_1} = \int_{UV} x_1 x_3 \, dV - L^2 \int_{US} (x_1 \nu_2 - x_2 \nu_1) \psi_1 \, dS$$

$$= \int_{UV} x_1 x_3 \, dV - L^2 \int_{US} \nu_i x_1 x_2 \frac{\partial \psi_1}{\partial x_i} \, dS + 2 L^2 \int_{US} x_2 \nu_1 \psi_1 \, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} x_1 x_3 \, dV - L^2 \int_{US} x_1 x_2 \frac{2 x_3 \nu_2}{L^2} \, dS + 2 L^2 \int_{US} x_2 \nu_1 \psi_1 \, dS \qquad \text{(from (2.33))}$$

$$= \int_{UV} x_1 x_3 \, dV - 2 \int_{UV} \frac{\partial}{\partial x_2} (x_1 x_2 x_3)\, dV + 2 L^2 \int_{US} x_2 \nu_1 \psi_1 \, dS \qquad \text{(Theorem I)}$$

$$= -\int_{UV} x_1 x_3 \, dV + 2 L^2 \int_{US} x_2 \nu_1 \psi_1 \, dS \qquad \text{(D. 7)}$$

Define $I_{31} = -\rho\, I_{3,\omega_1}$.

Consider the integral of the fifth term

$$I_{i,\omega_2} = \int_{UV} \left| \epsilon_{ij3}\, x_j\, x_1 + \epsilon_{ij1}\, x_j\, x_3 \right| dV - L^2 \int_{UV} \epsilon_{ijk}\, x_j \frac{\partial \psi_2}{\partial x_k}\, dV$$

$$= \int_{UV} \left| \epsilon_{ij3}\, x_j\, x_1 + \epsilon_{ij1}\, x_j\, x_3 \right| dV - L^2 \int_{US} \epsilon_{ijk} x_j \psi_2\, \nu_k\, dS \qquad \text{(Theorem V)}$$

Thus

$$I_{1,\omega_2} = \int_{UV} x_1\, x_2\, dV - L^2 \int_{US} (x_2\, \nu_3 - x_3\, \nu_2)\, \psi_2\, dS$$

$$= \int_{UV} x_1\, x_2\, dV - L^2 \int_{US} \nu_i\, x_2\, x_3 \frac{\partial \psi_2}{\partial x_i}\, dS + 2\, L^2 \int_{US} x_3\, \nu_2 \psi_2\, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} x_1\, x_2\, dV - L^2 \int_{US} x_2\, x_3 \frac{2\, x_1\, \nu_3}{L^2}\, dS + 2\, L^2 \int_{US} x_3 \nu_2 \psi_2\, dS \qquad \text{(from (2.34))}$$

$$= \int_{UV} x_1\, x_2\, dV - 2 \int_{US} \frac{\partial}{\partial x_3} (x_1\, x_2\, x_3)\, dS + 2\, L^2 \int_{US} x_3\, \nu_2 \psi_2\, dS \qquad \text{(Theorem I)}$$

$$= - \int_{US} x_1\, x_2\, dV + 2\, L^2 \int_{US} x_3 \nu_2 \psi_2\, dS \qquad \text{(D. 8)}$$

Define $I_{12} = -\rho\, I_{1,\omega_2}$. Similarly

$$I_{2,\omega_2} = \int_{UV} (-x_1^2 + x_3^2)\, dV - L^2 \int_{US} (x_3 \nu_1 - x_1 \nu_3)\, \psi_2\, dS$$

$$= \int_{UV} (-x_1^2 + x_3^2)\, dV - L^2 \int_{US} \nu_i\, x_1\, x_3 \frac{\partial \psi_2}{\partial x_i}\, dS + 2\, L^2 \int_{US} x_1\, \nu_3\, \psi_2\, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} (-x_1^2 + x_3^2)\, dV - L^2 \int_{US} x_1\, x_3 \frac{2\, x_1\, \nu_3}{L^2}\, dS + 2\, L^2 \int_{US} x_1\, \nu_3\, \psi_2\, dS \qquad \text{(from (2.34))}$$

$$= \int_{UV} (-x_1^2 + x_3^2)\, dV - 2 \int_{UV} \frac{\partial}{\partial x_3} (x_1^2\, x_3)\, dV + 2\, L^2 \int_{US} x_1\, \nu_3\, \psi_2\, dS \qquad \text{(Theorem I)}$$

$$= \int_{UV} (x_1^2 + x_3^2)\, dV - 4 \int_{UV} x_1^2\, dV + 2\, L^2 \int_{US} x_1\, \nu_3\, \psi_2\, dS \qquad \text{(D. 9)}$$

Define $I_{22} = \rho I_{2,\omega_2}$. Finally

$$I_{3,\omega_2} = \int_{UV} - x_2 x_3 \, dV - L^2 \int_{US} (x_1 \nu_2 - x_2 \nu_1) \psi_2 \, dS$$

$$= \int_{UV} - x_2 x_3 \, dV - L^2 \int_{US} \nu_i x_1 x_2 \frac{\partial \psi_2}{\partial x_i} dS + 2 L^2 \int_{US} x_2 \nu_1 \psi_2 \, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} - x_2 x_3 \, dV - L^2 \int_{US} x_1 x_2 \frac{2 x_1 \nu_3}{L^2} dS + 2 L^2 \int_{US} x_2 \nu_1 \psi_2 \, dS \qquad \text{(from (2.34))}$$

$$= - \int_{UV} x_2 x_3 \, dV + 2 L^2 \int_{US} x_2 \nu_1 \psi_2 \, dS \qquad \text{(Theorem I)} \qquad \text{(D.10)}$$

Define $I_{32} = -\rho I_{3,\omega_2}$.

Consider the integral of the sixth term

$$I_{i,\omega_3} = \int_{UV} (\epsilon_{ij2} x_j x_1 + \epsilon_{ij1} x_j x_2) \, dV - L^2 \int_{UV} \epsilon_{ijk} x_j \frac{\partial \psi_3}{\partial x_k} dV$$

$$= \int_{UV} (\epsilon_{ij2} x_j x_1 + \epsilon_{ij1} x_j x_2) \, dV - L^2 \int_{US} \epsilon_{ijk} x_j \psi_3 \nu_k \, dS \qquad \text{(Theorem V)}$$

Thus

$$I_{1,\omega_3} = \int_{UV} - x_3 x_1 \, dV - L^2 \int_{US} (x_2 \nu_3 - x_3 \nu_2) \psi_3 \, dS$$

$$= \int_{UV} - x_3 x_1 \, dV - L^2 \int_{US} \nu_i x_2 x_3 \frac{\partial \psi_3}{\partial x_i} dS + 2 L^2 \int_{US} x_3 \nu_2 \psi_3 \, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} - x_3 x_1 \, dV - L^2 \int_{US} x_2 x_3 \frac{2 x_2 \nu_1}{L^2} dS + 2 L^2 \int_{US} x_3 \nu_2 \psi_3 \, dS \qquad \text{(from (2.35))}$$

$$= - \int_{UV} x_3 x_1 \, dV + 2 L^2 \int_{US} x_3 \nu_2 \psi_3 \, dS \qquad \text{(Theorem I)}$$

Define $I_{13} = -\rho I_{1,\omega_3}$. Similarly

$$I_{2,\omega_3} = \int_{UV} x_3 x_2 \, dV - L^2 \int_{US} (x_3 \nu_1 - x_1 \nu_3) \psi_3 \, dS$$

$$I_{2,\omega_3} = \int_{UV} x_3 x_2 \, dV - L^2 \int_{US} \nu_i x_1 x_3 \frac{\partial \psi_3}{\partial x_i} dS + 2L^2 \int_{US} x_1 \nu_3 \psi_3 \, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} x_3 x_2 \, dV - L^2 \int_{US} x_1 x_3 \frac{2 x_2 \nu_1}{L^2} dS + 2L^2 \int_{US} x_1 \nu_3 \psi_3 \, dS \qquad \text{(from (2.35))}$$

$$= \int_{UV} x_3 x_2 \, dV - 2 \int_{UV} \frac{\partial}{\partial x_1} (x_1 x_2 x_3) \, dV + 2L^2 \int_{US} x_1 \nu_3 \psi_3 \, dS \qquad \text{(Theorem I)}$$

$$= - \int_{UV} x_2 x_3 \, dV + 2L^2 \int_{US} x_1 \nu_3 \psi_3 \, dS$$

Define $I_{23} = -\rho I_{2,\omega_3}$. Finally

$$I_{3,\omega_3} = \int_{UV} (x_1^2 - x_2^2) \, dV - L^2 \int_{US} (x_1 \nu_2 - x_2 \nu_1) \psi_3 \, dS$$

$$= \int_{UV} (x_1^2 - x_2^2) \, dV - L^2 \int_{US} \nu_i x_1 x_2 \frac{\partial \psi_3}{\partial x_i} dS + 2L^2 \int x_2 \nu_1 \psi_3 \, dS \qquad \text{(Theorem IV)}$$

$$= \int_{UV} (x_1^2 - x_2^2) \, dV - L^2 \int_{US} x_1 x_2 \frac{2 x_2 \nu_1}{L^2} dS + 2L^2 \int x_2 \nu_1 \psi_3 \, dS \qquad \text{(from (2.35))}$$

$$= \int_{UV} (x_1^2 - x_2^2) \, dV - 2 \int_{UV} \frac{\partial}{\partial x_1} (x_1 x_2^2) \, dV + 2L^2 \int x_2 \nu_1 \psi_3 \, dS \qquad \text{(Theorem I)}$$

$$= \int_{UV} (x_1^2 + x_2^2) \, dV - 4 \int_{UV} x_2^2 \, dV + 2L^2 \int_{US} x_2 \nu_1 \psi_3 \, dS$$

Define $I_{33} = \rho I_{3,\omega_3}$.

Consider the integral of the last term, and assume that the order of integration and summation may be interchanged.

$$I_{i,mn} = \int_{UV} \epsilon_{ijk} x_j \frac{\partial \phi_{mn}}{\partial x_k} dV = \int_{UV} \epsilon_{ijk} \frac{\partial}{\partial x_k} (x_j \phi_{mn}) \, dV$$

$$= \int_{US} \epsilon_{ijk} x_j \phi_{mn} \nu_k \, dS \qquad \text{(Theorem V)}$$

Thus

$$I_{1,mn} = \int_{US} (x_2\nu_3 - x_3\nu_2)\ \phi_{mn}\ dS$$

$$= \int_{US} \nu_i x_2 x_3 \frac{\partial \phi_{mn}}{\partial x_i}\ dS - 2\int_{US} x_3 \nu_2 \phi_{mn}\ dS \qquad \text{(Theorem IV)}$$

$$= V\ K_{mn}\ \gamma_{mn}\ (b_{mn} - h_{mn}) \qquad \text{((2.47) and (B.11))}$$

Similarly

$$I_{2,mn} = \int_{US} (x_3\nu_1 - x_1\nu_3)\ \phi_{mn}\ dS$$

$$= \int_{US} \nu_i x_1 x_3 \frac{\partial \phi_{mn}}{\partial x_i}\ dS - 2\int_{US} x_1 \nu_3\ \phi_{mn}\ dS \qquad \text{(Theorem IV)}$$

$$= V\ K_{mn}\ \gamma_{mn}\ (a_{mn} - d_{mn}) \qquad \text{((2.47) and (B.11))}$$

Finally

$$I_{3,mn} = \int_{US} (x_1\nu_2 - x_2\nu_1)\ dS$$

$$= \int_{US} \nu_i x_1 x_2 \frac{\partial \phi_{mn}}{\partial x_i}\ dS - 2\int_{US} x_2 \nu_1\ \phi_{mn}\ dS \qquad \text{(Theorem IV)}$$

$$= V\ K_{mn}\ \gamma_{mn}\ (e_{mn} - f_{mn}) \qquad \text{((2.47) and (B.11))}$$

Combining the values of the various integrals into (D.1) yields

$$T_i = -\dot{\omega}_1 \left| I_{11}\delta_{i1} - I_{21}\delta_{i2} - I_{31}\delta_{i3} \right| - \dot{\omega}_2 \left| -I_{12}\delta_{i1} + I_{22}\delta_{i2} - I_{32}\delta_{i3} \right|$$

$$- \dot{\omega}_3 \left| -I_{13}\delta_{i1} - I_{23}\delta_{i2} + I_{33}\delta_{i3} \right| - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} \left| \left| L\,(b_{mn} - h_{mn})\, \ddot{\xi}_{mn} \right.\right.$$

$$\left. + \alpha_3 b_{mn}\, \xi_{mn} \right| \delta_{i1} + \left| L\,(a_{mn} - d_{mn}) \ddot{\xi}_{mn} - \alpha_3\, a_{mn}\, \xi_{mn} \right| \delta_{i2}$$

$$\left. + \left| L\,(e_{mn} - f_{mn})\ \ddot{\xi}_{mn} \right| \delta_{i3} \right|$$

APPENDIX E

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The substitution of (3.34) into (3.35) yields

$$F_i' = \rho \int_V \Bigg\{ -\alpha_3 \delta_{i3} - \alpha_2 \delta_{i2} - \dot{\omega}_1 \left[\delta_{i2} (x_3' - 2 L_1) + x_2' \delta_{i3} - L^2 \frac{\partial \psi_1 (x_1', x_2', x_3' - L_1)}{\partial x_i'} \right]$$

$$- L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial \phi_{mn} (x_1', x_2', x_3' - L_1)}{K_{mn} \, \partial x_i'} \ddot{\xi}_{mn} \Bigg\} dV$$

$$= -\rho \alpha_3 V \delta_{i3} - \alpha_2 \rho V \delta_{i2} - \rho \dot{\omega}_1 \int_V \Bigg[\delta_{i2} (x_3' - 2 L_1) + x_2' \delta_{i2}$$

$$- L^2 \frac{\partial \psi_1' (x_1', x_2', x_3' - L_1)}{\partial x_i'} \Bigg] dV$$

$$-\rho L \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\xi}_{mn} \frac{1}{K_{mn}} \int_V \frac{\partial \phi_{mn} (x_1', x_2', x_3' - L_1)}{\partial x_i'} \, dV \qquad \text{(E.1)}$$

As in the previous appendices, integrals will be taken over the undisturbed configuration if a multiplicative factor which is assumed to be small occurs. Consider

$$\rho \dot{\omega}_1 \int_V \left[\delta_{i2} (x_3' - 2 L_1) + x_2' \delta_{i2} - L^2 \frac{\partial \psi_1' (x_1', x_2', x_3' - L_1)}{\partial x_i'} \right] dV$$

$$= \dot{\omega}_1 \Bigg[-2 L_1 \rho V \delta_{i2} + \rho \delta_{i2} \int_V x_3' \, dV + \rho \delta_{i2} \int_V x_2' \, dV$$

$$- \rho L^2 \int_V \frac{\partial \psi_1' (x_1', x_2', x_3' - L_1)}{\partial x_i'} \Bigg] dV$$

Now

$$\int_{UV} x_3' \, dV = \int_{UV} (x_3 + L_1) \, dV = L_1 V$$

$$\int_{UV} x_2' \, dV = \int_{UV} x_2 \, dV = 0$$

$$\int_{UV} \frac{\partial \psi_1'}{\partial x_i'}\, dV = \int_{UV} \frac{\partial \psi_1}{\partial x_i}\, dV = 0 \qquad \text{by (C.5)}$$

and

$$\int_{UV} \frac{\partial \phi_{mn}(x_1', x_2', x_3' - L_1)}{\partial x_i'}\, dV = \int_{UV} \frac{\partial \phi_{mn}(x_1, x_2, x_3)}{\partial x_i}\, dV$$

$$= \frac{K_{mn} V \gamma_{mn}}{L} \left[a_{mn}\, \delta_{i1} + b_{mn}\, \delta_{i2} \right]$$

Thus

$$F_i' = -\alpha_3 M \delta_{i3} - \alpha_2 M \delta_{i2} + \dot{\omega}_1 L_1 M \delta_{i2} - M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \ddot{\xi}_{mn} \gamma_{mn} \left[a_{mn} \delta_{i1} \right.$$
$$\left. + b_{mn}\, \delta_{i2} \right] \tag{E.2}$$

The moments are given by

$$T_i' = \rho \int_V \epsilon_{ijk}\, x_j' \left\{ -\alpha_3 \delta_{k3} - \alpha_2 \delta_{k2} - \dot{\omega}_1 \left[\delta_{k2} (x_3' - 2 L_1) + x_2' \delta_{k3} \right. \right.$$
$$\left. \left. - L^2 \frac{\partial \psi_1'}{\partial x_k'} \right] - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial \phi_{mn}'}{\partial x_k'} \frac{1}{K_{mn}} \ddot{\xi}_{mn} \right\} dV \tag{E.3}$$

The integrals will be taken over the undisturbed configuration if a multiplicative factor which is assumed to be small occurs. The integral of each term will be considered separately.

Consider the integral of the first term

$$-\rho \alpha_3 \int_V \epsilon_{ij3}\, x_j'\, dV = -\rho \alpha_3 \int_V \epsilon_{ijk} \frac{\partial (x_j' x_3')}{\partial x_k'}\, dV \tag{E.4}$$

$$= -\rho \alpha_3 \int_S \epsilon_{ijk} x_j' x_3' \nu_k' dS = -\rho \alpha_3 \left[\int_{US} \epsilon_{ijk} x_j' x_3' \nu_k' dS \right.$$
$$\left. + \int_{S-US} \epsilon_{ijk} x_j' x_3' \nu_k' dS \right]$$

But

$$\int_{US} \epsilon_{ijk}\, x_j'\, x_3'\, \nu_k'\, dS = \int_{UV} \epsilon_{ijk}\, x_j'\, \delta_{k3}\, dV = \int_{UV} \epsilon_{ij3}\, x_j\, dV = 0$$

$$-\rho\,\alpha_3 \int_{S-US} \epsilon_{ijk}\, x_j'\, x_3'\, \nu_k'\, dS \approx -\rho\,\alpha_3 \left[\int_{FS} \epsilon_{ijk}\, x_j'\, x_3'\, \nu_k'\, dS - \int_{UFS} \epsilon_{ijk}\, x_j'\, (L + L_1)\nu_k'\, dS\right]$$

$$= -\rho\,\alpha_3 \int_{FS} \epsilon_{ijk}\, x_j'(L_1 + L + \eta)\nu_k'\, dS + \rho\,\alpha_3 \int_{UFS} \epsilon_{ijk}\, x_j'\, (L + L_1)\nu_k'\, dS$$

$$= -\rho\,\alpha_3 \int_{FS} \epsilon_{ijk}\, x_j'\, \eta\, \nu_k'\, dS$$

$$\approx -\rho\,\alpha_3 \int_{UFS} \epsilon_{ij3}\, x_j'\, \eta\, dS$$

$$= -\rho\,\alpha_3 \int_{UFS} \epsilon_{ij3}\, x_j\, \eta\, dS$$

$$= -\alpha_3\, M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn}\, \gamma_{mn} \left[b_{mn}\,\delta_{i1} - a_{mn}\,\delta_{i2}\right] \quad (E.5)$$

Consider the integral of the second term

$$-\rho\,\alpha_2 \int_{UV} \epsilon_{ij2}\, x_j'\, dV = -\rho\,\alpha_2 \int_{UV} \epsilon_{ij2}\, x_j\, dV - \rho\,\alpha_2 \int_{UV} \epsilon_{i32}\, L_1\, dV$$

$$= L_1\, M\, \alpha_2\, \delta_{i1} \quad (E.6)$$

Consider the integral of the third term

$$-\rho\,\dot{\omega}_1 \int_{UV} \left\{\frac{\partial}{\partial x_k'} (x_2')(x_3' - 2\,L_1) - L^2 \frac{\partial}{\partial x_k'} \psi_1\,(x_1', x_2', x_3' - L_1)\right\} \epsilon_{ijk}\, x_j'\, dV$$

$$= -\rho\,\dot{\omega}_1 \left\{\int_{UV} \epsilon_{ijk}\, x_j' \left[(x_3' - 2\,L_1)\,\delta_{k2} + x_2'\,\delta_{k3}\right] dV - L^2 \int_{US} \epsilon_{ijk}\, x_j'\, \nu_k'\, \psi_1'\, dS\right\} \quad (E.7)$$

For i = 1, (E.7) becomes

$$-\rho\,\dot{\omega}_1 \left\{ \int_{UV} \left[\epsilon_{132}\, x_3' (x_3' - 2\,L_1) + \epsilon_{123}\, x_2' x_2' \right] dV - L^2 \int_{US} (x_2' \nu_3' - x_3' \nu_2')\, \psi_1'\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ \int_{UV} \left[x_2^2 - (x_3 + L_1)(x_3 - L_1) \right] dV - L^2 \int_{US} \nu_k' x_2' x_3' \frac{\partial \psi_1'}{\partial x_k'}\, dS \right.$$

$$\left. + 2\,L^2 \int_{US} x_3' \nu_2' \psi_1'\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ \int_{UV} \left(x_2^2 + x_3^2 \right) dV - 2 \int_{UV} x_3^2\, dV + L_1^2 V - L^2 \int_{US} \nu_k x_2 x_3 \frac{\partial \psi_1}{\partial x_k}\, dS \right.$$

$$\left. - L^2 L_1 \int_{US} \nu_k x_2 \frac{\partial \psi_1}{\partial x_k}\, dS + 2\,L^2 \int_{US} x_3 \nu_2 \psi_1\, dS + 2\,L^2 L_1 \int_{US} \nu_2 \psi_1\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ \int_{UV} \left(x_2^2 + x_3^2 \right) dV - 2 \int_{UV} x_3^2\, dV + L_1^2 V - 2 \int_{UV} \frac{\partial}{\partial x_2} \left(x_2 x_3^2 \right) dV \right.$$

$$\left. - L^2 L_1 \int_{US} x_2 \frac{2\, x_3 \nu_2}{L^2}\, dS + 2\,L^2 \int_{US} x_3 \nu_2 \psi_1\, dS + 2\,L^2 L_1 \int_{UV} x_2 \nu_k \frac{\partial \psi_1}{\partial x_k}\, dV \right\}$$

$$= -\dot{\omega}_1 \left\{ \rho \int_{UV} \left(x_2^2 + x_3^2 \right) dV - 4\,\rho \int_{UV} x_3^2\, dV + L_1^2 M + 2\,\rho\, L^2 \int_{US} x_3 \nu_2 \psi_1\, dS \right\} \qquad \text{(E.8)}$$

For i = 2, (E.7) becomes

$$-\rho\,\dot{\omega}_1 \left\{ \int_{UV} \epsilon_{213}\, x_1' x_2'\, dV - L^2 \int_{US} (x_3' \nu_1' - x_1' \nu_3')\, \psi_1'\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ - \int_{UV} x_1 x_2\, dV - L^2 \int_{US} \nu_k' x_3' x_1' \frac{\partial \psi_1'}{\partial x_k'}\, dS + 2\,L^2 \int_{US} x_1' \nu_3' \psi_1'\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ - \int_{UV} x_1 x_2\, dV - L^2 \int_{US} \nu_k x_3 x_1 \frac{\partial \psi_1}{\partial x_k}\, dS \right.$$

$$\left. - L^2 \int_{US} \nu_k L_1 \frac{\partial \psi_1}{\partial x_k}\, dS + 2\,L^2 \int_{US} x_1 \nu_3 \psi_1\, dS \right\}$$

Use the boundary conditions on ψ_1 to write this expression as

$$= -\rho\,\dot{\omega}_1 \left\{ - \int_{UV} x_1\, x_2\, dV - 2 \int_{US} x_3^2\, x_1\, \nu_2\, dS - 2\, L_1 \int_{US} x_3\, \nu_2\, dS \right.$$

$$\left. + 2\, L^2 \int_{US} x_1\, \nu_3\, \psi_1\, dS \right\}$$

Apply Theorem I to get

$$= -\dot{\omega}_1 \left\{ -\rho \int_{UV} x_1\, x_2\, dV + 2\rho\, L^2 \int_{US} x_1\, \nu_3\, \psi_1\, dS \right\} \tag{E.9}$$

For i = 3, (E.7) becomes

$$-\rho\,\dot{\omega}_1 \left\{ \int_{UV} \epsilon_{312}\, x_1' (x_3' - 2\, L_1)\, dV - L^2 \int_{US} (x_1' \nu_2' - x_2' \nu_1')\, \psi_1'\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ \int_{UV} x_1\, (x_3 - L_1)\, dV - L^2 \int_{US} (x_1\, \nu_2 - x_2\, \nu_1)\, \psi_1\, dS \right\}$$

$$= -\rho\,\dot{\omega}_1 \left\{ - \int_{UV} x_1\, x_3\, dV + 2\, L^2 \int_{US} x_2\, \nu_1\, \psi_1\, dS \right\} \tag{E.10}$$

Consider the last integral in (E.3)

$$\int_{UV} \frac{\partial \phi_{mn} (x_1', x_2', x_3' - L_1)}{\partial x_k'}\, \epsilon_{ijk}\, x_j'\, dV = \int_{US} \epsilon_{ijk}\, x_j'\, \phi_{mn}'\, \nu_k'\, dS \tag{E.11}$$

For i = 1, (E.11) becomes

$$\int_{US} \epsilon_{1jk}\, x_j\, \phi_{mn}\, \nu_k\, dS + \int_{US} \epsilon_{132}\, L_1\, \phi_{mn}\, \nu_2\, dS$$

$$= \int_{US} \epsilon_{1jk}\, x_j\, \phi_{mn}\, \nu_k\, dS - L_1 \int_{US} \nu_k\, x_2\, \frac{\partial \phi_{mn}}{\partial x_k}\, dS$$

$$= \int_{US} \epsilon_{1jk}\, x_j\, \phi_{mn}\, \nu_k\, dS - \frac{L_1\, K_{mn}}{L} \int_{UFS} x_2\, \phi_{mn}\, dS$$

$$= V\, K_{mn}\, \gamma_{mn} \left[b_{mn} - h_{mn} \right] - \frac{L_1}{L}\, V\, K_{mn}\, b_{mn}\, \gamma_{mn} \tag{E.12}$$

For $i = 2$, (E.11) becomes

$$\int_{US} \epsilon_{2jk} x_j \phi_{mn} \nu_k \, dS + \int_{US} \epsilon_{231} L_1 \phi_{mn} \nu_1 \, dS$$

$$= \int_{US} \epsilon_{2jk} x_j \phi_{mn} \nu_k \, dS + L_1 \int_{US} \phi_{mn} \nu_1 \, dS$$

$$= \int_{US} \epsilon_{2jk} x_j \phi_{mn} \nu_k \, dS + L_1 \int_{US} x_1 \nu_k \frac{\partial \phi_{mn}}{\partial x_k} \, dS$$

$$= V K_{mn} \gamma_{mn} (a_{mn} - d_{mn}) + \frac{L_1 K_{mn}}{L} V a_{mn} \gamma_{mn} \qquad \text{(E.13)}$$

For $i = 3$, (E.11) becomes

$$\int_{US} \epsilon_{3jk} x_j \phi_{mn} \nu_k \, dS = V K_{mn} \gamma_{mn} (e_{mn} - f_{mn}) \qquad \text{(E.14)}$$

Combining the previous results in (E.3) yields

$$T_i' = M L_1 \alpha_2 \delta_{i1} - \dot{\omega}_1 \left[I_{11}' \delta_{i1} - I_{21}' \delta_{i2} - I_{31}' \delta_{i3} \right]$$

$$- M \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} \left\{ \left[\left\{ L (b_{mn} - h_{mn}) - L_1 b_{mn} \right\} \ddot{\xi}_{mn} + \alpha_3 b_{mn} \xi_{mn} \right] \delta_{i1} \right.$$

$$+ \left[\left\{ L (a_{mn} - d_{mn}) + L_1 a_{mn} \right\} \ddot{\xi}_{mn} - \alpha_3 a_{mn} \xi_{mn} \right] \delta_{i2}$$

$$\left. + \left[L (e_{mn} - f_{mn}) \ddot{\xi}_{mn} \right] \delta_{i3} \right\} \qquad \text{(E.15)}$$

where

$$\left.\begin{aligned} I_{11}' &= \rho \int_{UV} \left(x_2^2 + x_3^2 \right) dV - 4 \rho \int_{UV} x_3^2 \, dV + 2 \rho L^2 \int_{US} x_3 \nu_2 \psi_1 \, dS + L_1^2 M \\ I_{21}' &= \rho \int_{UV} x_1 x_2 \, dV - 2 \rho L^2 \int_{US} x_1 \nu_3 \psi_1 \, dS \\ I_{31}' &= \rho \int_{UV} x_1 x_3 \, dV - 2 \rho L^2 \int_{US} x_2 \nu_1 \psi_1 \, dS \end{aligned}\right\} \qquad \text{(E.16)}$$

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